

RELATIVE SINGULARITY CATEGORIES I: AUSLANDER RESOLUTIONS

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Dedicated to Idun Reiten on the occasion of her 70th birthday

ABSTRACT. We study the relations between two triangulated categories associated with a Gorenstein singularity R : the singularity category and the relative singularity category, which was associated with a (non-commutative) resolution A of R in joint work of Burban and the first author. We show that if R has only finitely many indecomposable maximal Cohen–Macaulay modules and A is the Auslander algebra of R , then these two categories mutually determine each other. Knörrer’s periodicity result yields a wealth of interesting and explicit examples: in particular, we study the case of ADE–singularities in all dimensions.

1. INTRODUCTION

Triangulated categories of singularities were introduced and studied by Buchweitz [22] and later also by Orlov [70, 71, 72] who related them to Kontsevich’s Homological Mirror Symmetry Conjecture. They may be seen as a categorical measure for the complexity of the singularities of a Noetherian scheme X . If X has only isolated Gorenstein singularities x_1, \dots, x_n , then the singularity category is triangle equivalent to the direct sum of the stable categories of maximal Cohen–Macaulay $\hat{\mathcal{O}}_{x_i}$ -modules (up to direct summands) [22, 72].

Starting with Van den Bergh’s works [87, 88], non-commutative analogues of (crepant) resolutions (NC(C)R) of singularities have been studied intensively in recent years. Non-commutative resolutions are useful even if the primary interest lies in commutative questions: for example, the Bondal-Orlov Conjecture concerning derived equivalences between (commutative) crepant resolutions and the derived McKay-Correspondence [20, 50] led Van den Bergh to the notion of a NCCR. Moreover, moduli spaces of quiver representations provide a very useful technique to obtain commutative resolutions from non-commutative resolutions, see e.g. [88, 90].

Inspired by the construction of the singularity category, Burban and the first author introduced and studied the notion of *relative singularity categories* [23]. These categories measure the difference between the derived category of a non-commutative resolution (NCR) [31] and the smooth part $K^b(\mathbf{proj} - R) \subseteq \mathcal{D}^b(\mathbf{mod} - R)$ of the derived category of the singularity. Continuing this line of investigations, this article focuses on the relation between relative and classical singularity categories.

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The techniques developed in this article led to a ‘purely commutative’ result: in joint work with Iyama and Wemyss [46], we decompose Iyama & Wemyss’ ‘new triangulated category’ for complete rational surface singularities [47] into blocks of singularity categories of ADE-singularities. Moreover, using relative singularity categories, Van den Bergh & Thanhoffer de Völcsy showed [30] that the stable category of a complete Gorenstein quotient singularity of Krull dimension three is a generalized cluster category [2, 39]. We recover some of their results using quite different techniques. Let us give a more detailed outline of the results of this article.

1.1. Setup. Let k be an algebraically closed field. Let (R, \mathfrak{m}) be a commutative local complete Gorenstein k -algebra such that $k \cong R/\mathfrak{m}$. Let $\text{MCM}(R) = \{M \in \text{mod } -R \mid \text{Ext}_R^i(M, R) = 0 \text{ for all } i > 0\}$ be the full subcategory of *maximal Cohen–Macaulay* R -modules. Let $M_0 = R, M_1, \dots, M_t$ be pairwise non-isomorphic indecomposable MCM R -modules and $A = \text{End}_R(M := \bigoplus_{i=0}^t M_i)$. If $\text{gldim}(A) < \infty$ then A is called a *non-commutative resolution* (NCR) of R (cf. [31]). For example, if R has only finitely many indecomposable MCMs and M denotes their direct sum, then the *Auslander algebra* $\text{Aus}(\text{MCM}(R)) := \text{End}_R(M)$ is a NCR ([44]). There is a fully faithful triangle functor $K^b(\text{proj } -R) \rightarrow \mathcal{D}^b(\text{mod } -A)$, whose essential image equals $\text{thick}(eA) \subseteq \mathcal{D}^b(\text{mod } -A)$ for some idempotent $e \in A$.

Definition 1.2. The *relative singularity category* is the Verdier quotient category

$$\Delta_R(A) := \frac{\mathcal{D}^b(\text{mod } -A)}{K^b(\text{proj } -R)} \cong \frac{\mathcal{D}^b(\text{mod } -A)}{\text{thick}(eA)}. \quad (1.1)$$

Definition 1.3. The *classical singularity category* is the Verdier quotient category

$$\mathcal{D}_{sg}(R) := \mathcal{D}^b(\text{mod } -R)/K^b(\text{proj } -R). \quad (1.2)$$

Buchweitz has shown that the singularity category $\mathcal{D}_{sg}(R)$ is triangle equivalent to the stable category of maximal Cohen–Macaulay modules $\underline{\text{MCM}}(R)$ [22].

1.4. Main Result. It is natural to ask how the two concepts of singularity categories defined above are related. Our main result gives a first answer to this question.

Theorem 5.19. *Let R and R' be MCM–representation finite complete Gorenstein k -algebras with Auslander algebras $A = \text{Aus}(\text{MCM}(R))$ and $A' = \text{Aus}(\text{MCM}(R'))$, respectively. Then the following statements are equivalent.*

- (i) *There is an equivalence $\underline{\text{MCM}}(R) \cong \underline{\text{MCM}}(R')$ of triangulated categories.*
- (ii) *There is an equivalence $\Delta_R(A) \cong \Delta_{R'}(A')$ of triangulated categories.*

The implication (ii) \Rightarrow (i) holds more generally for non-commutative resolutions A and A' of arbitrary isolated Gorenstein singularities R and R' , respectively.

Knörrer’s periodicity theorem [58, 85] yields a wealth of non-trivial examples for (i):

$$\underline{\text{MCM}}(S/(f)) \xrightarrow{\sim} \underline{\text{MCM}}(S[[x, y]]/(f + xy)), \quad (1.3)$$

where $S = k[[z_0, \dots, z_d]]$, f is a non-zero element in (z_0, \dots, z_d) and $d \geq 0$.

Example 1.5. Let $R = \mathbb{C}[[x]]/(x^2)$ and $R' = \mathbb{C}[[x, y, z]]/(x^2 + yz)$. Knörrer's equivalence (1.3) in conjunction with Theorem 5.19 above, yields a triangle equivalence $\Delta_R(\text{Aus}(\text{MCM}(R))) \cong \Delta_{R'}(\text{Aus}(\text{MCM}(R')))$, which may be written explicitly as

$$\frac{\mathcal{D}^b \left(1 \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{i} \end{array} 2 \Big/ (pi) \right)}{K^b(\text{add } P_1)} \xrightarrow{\sim} \frac{\mathcal{D}^b \left(1 \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \\ \xleftarrow{y} \\ \xleftarrow{x} \end{array} 2 \Big/ (xy - yx) \right)}{K^b(\text{add } P_1)}. \quad (1.4)$$

The quiver algebra on the right is the completion of the preprojective algebra of the Kronecker quiver $\circ \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \circ$. Moreover, the derived McKay–Correspondence [50, 20] shows that this algebra is derived equivalent to the minimal resolution of the completion of the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$.

1.6. Idea of the proof. We prove Theorem 5.19 by developing a general dg algebra framework. More precisely, to every Hom-finite idempotent complete algebraic triangulated category \mathcal{T} with finitely many indecomposable objects satisfying a certain extra condition (e.g. this holds for $\mathcal{T} = \underline{\text{MCM}}(R)$), we associate a dg algebra $\Lambda_{dg}(\mathcal{T})$ called the *dg Auslander algebra of \mathcal{T}* (Definition 4.5). It is completely determined by the triangulated category \mathcal{T} . Now, using recollements generated by idempotents, Koszul duality and the fractional Calabi–Yau property (1.8), we prove the existence of an equivalence of triangulated categories (Theorem 4.7)

$$\Delta_R(\text{Aus}(\text{MCM}(R))) \cong \text{per}(\Lambda_{dg}(\underline{\text{MCM}}(R))). \quad (1.5)$$

In particular, this shows that (i) implies (ii). Conversely, written in this language, the quotient functor (1.7), induces an equivalence of triangulated categories

$$\frac{\text{per}(\Lambda_{dg}(\underline{\text{MCM}}(R)))}{\mathcal{D}_{fd}(\Lambda_{dg}(\underline{\text{MCM}}(R)))} \longrightarrow \underline{\text{MCM}}(R). \quad (1.6)$$

Since the category $\mathcal{D}_{fd}(\Lambda_{dg}(\underline{\text{MCM}}(R)))$ of dg modules with finite dimensional total cohomology admits an intrinsic characterization in $\text{per}(\Lambda_{dg}(\underline{\text{MCM}}(R)))$, this proves that $\underline{\text{MCM}}(R)$ is determined by $\Delta_R(\text{Aus}(\text{MCM}(R)))$. Hence, (ii) implies (i).

Example. Let $R = \mathbb{C}[[z_0, \dots, z_d]]/(z_0^{n+1} + z_1^2 + \dots + z_d^2)$ be an A_n -singularity of even Krull dimension. Then the graded quiver Q of $\Lambda_{dg}(\underline{\text{MCM}}(R))$ is given as

$$\begin{array}{ccccccc} 1 & \xrightarrow{\alpha_1} & 2 & \xrightarrow{\alpha_2} & 3 & \xrightarrow{\alpha_3} & \dots & \xrightarrow{\alpha_{n-2}} & n-1 & \xrightarrow{\alpha_{n-1}} & n \\ \circlearrowleft_{\rho_1} & \alpha_1^* & \circlearrowleft_{\rho_2} & \alpha_2^* & \circlearrowleft_{\rho_3} & \alpha_3^* & & \alpha_{n-2}^* & \circlearrowleft_{\rho_{n-1}} & \alpha_{n-1}^* & \circlearrowleft_{\rho_n} \end{array}$$

where the broken arrows are concentrated in degree -1 and the remaining generators, i.e. solid arrows and idempotents, are in degree 0 . The continuous k -linear differential $d: \widehat{kQ} \rightarrow \widehat{kQ}$ is completely specified by sending ρ_i to the mesh relation (or preprojective relation) starting in the vertex i , e.g. $d(\rho_2) = \alpha_1\alpha_1^* + \alpha_2^*\alpha_2$.

We include a complete list of (the graded quivers, which completely determine) the dg Auslander algebras for ADE-singularities in all Krull dimensions in the Appendix.

Remark 1.7. The triangle equivalence (1.6) and its proof yield relations to generalized cluster categories [2, 39, 30] and stable categories of special Cohen-Macaulay modules over complete rational surface singularities [91, 47, 46]. Moreover, Bridgeland determined a connected component of the stability manifold of $\mathcal{D}_{fd}(\Lambda_{dg}(\underline{\mathbf{MCM}}(R)))$ for ADE-surfaces R [19]. We refer to Section 6 for more details on these remarks.

1.8. General properties of relative singularity categories. In the notations of the setup given in Subsection 1.1 above, we assume that R has an *isolated* singularity and that A is a NCR of R . Let $\underline{A} := A/AeA \cong \underline{\mathbf{End}}_R(M)$ be the corresponding stable endomorphism algebra. Since R is an isolated singularity, \underline{A} is a finite dimensional k -algebra. We denote the simple \underline{A} -modules by S_1, \dots, S_t . Then the relative singularity category $\Delta_R(A) = \mathcal{D}^b(\mathbf{mod} - A)/K^b(\mathbf{proj} - R)$ has the following properties:

- (a) All morphism spaces are finite dimensional over k (see [30] or Prop. 5.14).
- (b) $\Delta_R(A)$ is idempotent complete and $K_0(\Delta_R(A)) \cong \mathbb{Z}^t$ (see [23, Thm. 3.2]).
- (c) There is an exact sequence of triangulated categories (see [30] or Prop. 5.12)

$$\mathbf{thick}(S_1, \dots, S_t) = \mathcal{D}_{\underline{A}}^b(\mathbf{mod} - A) \longrightarrow \Delta_R(A) \longrightarrow \mathcal{D}_{sg}(R), \quad (1.7)$$

where $\mathcal{D}_{\underline{A}}^b(\mathbf{mod} - A) \subseteq \mathcal{D}^b(\mathbf{mod} - A)$ denotes the full subcategory consisting of complexes with cohomologies in $\mathbf{mod} - \underline{A}$. Moreover, this subcategory admits an intrinsic description inside $\mathcal{D}^b(\mathbf{mod} - A)$ (see [30] or Cor. 5.17).

- (d) If $\mathbf{add} M$ has d -almost split sequences [45], then $\mathcal{D}_{\underline{A}}^b(\mathbf{mod} - A)$ has a Serre functor ν , whose action on the generators S_i is given by

$$\nu^n(S_i) \cong S_i[n(d+1)], \quad (1.8)$$

where $n = n(S_i)$ is given by the length of the τ_d -orbit of M_i (Thm. 4.3).

- (e) Let $(\mathcal{D}_{\underline{A}}(\mathbf{Mod} - A))^c \subseteq \mathcal{D}_{\underline{A}}^b(\mathbf{Mod} - A)$ be the full subcategory of compact objects. There is an equivalence of triangulated categories (see Rem. 2.19).

$$\Delta_R(A) \cong (\mathcal{D}_{\underline{A}}(\mathbf{Mod} - A))^c. \quad (1.9)$$

- (f) Let M_{t+1}, \dots, M_s be further indecomposable $\mathbf{MCM} R$ -modules and let $A' = \mathbf{End}_R(\bigoplus_{i=0}^s M_i)$. There exists a fully faithful triangle functor (Prop. 5.10)

$$\Delta_R(A) \longrightarrow \Delta_R(A'). \quad (1.10)$$

- (g) If $\mathbf{kr.dim} R = 3$ and $\mathbf{MCM}(R)$ has a cluster-tilting object M , then $C = \mathbf{End}_R(M)$ is a *non-commutative crepant resolution* of R , see [44, Section 5]. If M' is another cluster-tilting object in $\mathbf{MCM}(R)$ and $C' = \mathbf{End}_R(M')$, then $? \otimes_C^L \mathbf{Hom}_R(M', M): \mathcal{D}^b(\mathbf{mod} C) \rightarrow \mathcal{D}^b(\mathbf{mod} C')$ is a triangle equivalence (see *loc. cit.* and [73, Prop. 4]), which is compatible with the embeddings from $K^b(\mathbf{proj} R)$ [73, Cor. 5]. Hence, one obtains a triangle equivalence

$$\Delta_R(C) \longrightarrow \Delta_R(C'). \quad (1.11)$$

Remark 1.9. The Hom-finiteness in (a) is surprising since (triangulated) quotient categories tend to behave quite poorly in this respect (see e.g. [23, Remark 6.5.]).

Remark 1.10. All our results actually hold in the generality of Gorenstein S -orders with an isolated singularity, in the sense of Auslander (see [10, Section III.1] and [11]) or finite dimensional selfinjective k -algebras. Here $S = (S, \mathfrak{n})$ denotes a local complete regular Noetherian k -algebra, with $k \cong S/\mathfrak{n}$. It is a matter of heavier notation and terminology to generalize our proofs to this setting.

1.11. Contents. Section 2 provides the necessary material on dg algebras and derived categories of dg modules. Moreover, we give an apparently new criterion for the Hom-finiteness of the category of perfect dg modules $\mathbf{per}(B)$ over some non-positive dg k -algebra B (Proposition 2.10). Further, we study recollements of derived categories of dg modules associated to an idempotent. Section 3 deals with equivalences between triangulated (quotient) categories arising from derived module categories of a right Noetherian ring A and idempotents in A . This is used in Section 5 to show that $\mathbf{MCM}(R)$ may be obtained as a triangle quotient of $\Delta_R(A)$. The first result in Section 4 shows that certain relative singularity categories enjoy a (weak) fractional Calabi–Yau property (Theorem 4.3). We use this and the results of Section 2 to show that for ‘finite’ Frobenius categories the relative Auslander singularity categories only depend on the stable category (Theorem 4.7). The results from Sections 2 to 4 are applied in Section 5 to study relative singularity categories over complete isolated Gorenstein k -algebras R . In particular, we prove our main result (Theorem 5.19). In Section 6, we remark on relations to Bridgeland’s stability manifold and generalized cluster categories. In the Appendix, we give a complete list of the dg Auslander algebras for the complex ADE-singularities in all Krull dimensions.

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2. DERIVED CATEGORIES

2.1. Notations. Let k be a commutative ring. Let $D = \mathbf{Hom}_k(?, k)$ denote the k -dual. When the input is a graded k -module, D means the graded dual. Namely, for $M = \bigoplus_{i \in \mathbb{Z}} M^i$, the graded dual DM has components $(DM)^i = \mathbf{Hom}_k(M^{-i}, k)$.

Generating subcategories/subsets. Let \mathcal{A} be an additive k -category. Let \mathcal{S} be a subcategory or a subset of objects of \mathcal{A} . We denote by $\mathbf{add}_{\mathcal{A}}(\mathcal{S})$ (respectively, $\mathbf{Add}_{\mathcal{A}}(\mathcal{S})$) the smallest full subcategory of \mathcal{A} which contains \mathcal{S} and which is closed under taking finite direct sums (respectively, all existing direct sums) and taking direct summands.

If \mathcal{A} is a triangulated category, then $\text{thick}_{\mathcal{A}}(\mathcal{S})$ (respectively, $\text{Tria}_{\mathcal{A}}(\mathcal{S})$) denotes the smallest triangulated subcategory of \mathcal{A} which contains \mathcal{S} and which is closed under taking direct summands (respectively, all existing direct sums).

When it does not cause confusion, we omit the subscripts and write the above notations as $\text{add}(\mathcal{S})$, $\text{Add}(\mathcal{S})$, $\text{thick}(\mathcal{S})$ and $\text{Tria}(\mathcal{S})$.

Derived categories of abelian categories. Let \mathcal{A} be an additive k -category. Let $*$ $\in \{\emptyset, -, +, b\}$ be a boundedness condition. Denote by $K^*(\mathcal{A})$ the homotopy category of complexes of objects in \mathcal{A} satisfying the boundedness condition $*$.

Let \mathcal{A} be an abelian k -category. Denote by $\mathcal{D}^*(\mathcal{A})$ the derived category of complexes of objects in \mathcal{A} satisfying the boundedness condition $*$.

Let R be a k -algebra. Without further remark, by an R -module we mean a right R -module. Denote by $\mathbf{Mod} - R$ the category of R -modules, and denote by $\mathbf{mod} - R$ (respectively, $\mathbf{proj} - R$) its full subcategory of finitely generated R -modules (respectively, finitely generated projective R -modules). When k is a field, we will also consider the category $\mathbf{fdmod} - R$ of those R -modules which are finite-dimensional over k . We often view $K^b(\mathbf{proj} - R)$ as a triangulated subcategory of $\mathcal{D}^*(\mathbf{Mod} - R)$.

Truncations. Let \mathcal{A} be an abelian k -category. For $i \in \mathbb{Z}$ and for a complex M of objects in \mathcal{A} , we define the *standard truncations* $\sigma^{\leq i}$ and $\sigma^{> i}$ by

$$(\sigma^{\leq i} M)^j = \begin{cases} M^j & \text{if } j < i, \\ \ker d_M^i & \text{if } j = i, \\ 0 & \text{if } j > i, \end{cases} \quad (\sigma^{> i} M)^j = \begin{cases} 0 & \text{if } j < i, \\ \frac{M^i}{\ker d_M^i} & \text{if } j = i, \\ M^j & \text{if } j > i, \end{cases}$$

and the *brutal truncations* $\beta_{\leq i}$ and $\beta_{\geq i}$ by

$$(\beta_{\leq i} M)^j = \begin{cases} M^j & \text{if } j \leq i, \\ 0 & \text{if } j > i, \end{cases} \quad (\beta_{\geq i} M)^j = \begin{cases} 0 & \text{if } j < i, \\ M^j & \text{if } j \geq i. \end{cases}$$

Their respective differentials are inherited from M . Notice that $\sigma^{\leq i}(M)$ and $\beta_{\geq i}(M)$ are subcomplexes of M and $\sigma^{> i}(M)$ and $\beta_{\leq i-1}(M)$ are the corresponding quotient complexes. Thus we have two sequences, which are componentwise short exact,

$$0 \rightarrow \sigma^{\leq i}(M) \rightarrow M \rightarrow \sigma^{> i}(M) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \beta_{\geq i}(M) \rightarrow M \rightarrow \beta_{\leq i-1}(M) \rightarrow 0.$$

Moreover, taking standard truncations behaves well with respect to cohomology.

$$H^j(\sigma^{\leq i} M) = \begin{cases} H^j(M) & \text{if } j \leq i, \\ 0 & \text{if } j > i, \end{cases} \quad H^j(\sigma^{> i} M) = \begin{cases} 0 & \text{if } j \leq i, \\ H^j(M) & \text{if } j > i. \end{cases}$$

2.2. DG algebras and their derived categories. Let A be a dg k -algebra. Consider the *derived category* $\mathcal{D}(A)$ of dg A -modules, see [51]. It is a triangulated category with shift functor being the shift of complexes [1]. It is obtained from the category $\mathcal{C}(A)$ of dg A -modules by formally inverting all quasi-isomorphisms. Let $\text{per}(A) = \text{thick}(A_A)$. A k -algebra R can be viewed as a dg algebra concentrated in

degree 0. In this case, we have $\mathcal{D}(R) = \mathcal{D}(\text{Mod} - R)$ and $\text{per}(R) = K^b(\text{proj} - R)$. Assume that k is a field. Consider the full subcategory $\mathcal{D}_{fd}(A)$ of $\mathcal{D}(A)$ consisting of those dg A -modules whose total cohomology is finite-dimensional.

Let M, N be dg A -modules. Define the complex $\mathcal{H}om_A(M, N)$ componentwise as

$$\mathcal{H}om_A^i(M, N) = \left\{ f \in \prod_{j \in \mathbb{Z}} \text{Hom}_k(M^j, N^{i+j}) \mid f(ma) = f(m)a \right\},$$

with differential given by $d(f) = d_N \circ f - (-1)^i f \circ d_M$ for $f \in \mathcal{H}om_A^i(M, N)$. The complex $\mathcal{E}nd_A(M) = \mathcal{H}om_A(M, M)$ with the composition of maps as product is a dg k -algebra. We will use the following results from [51]. Let A and B be dg k -algebras.

- Every dg A -module M has a natural structure of dg $\mathcal{E}nd_A(M)$ - A -bimodule.
- If M is a dg A - B -bimodule, then there is an adjoint pair of triangle functors

$$\mathcal{D}(A) \begin{array}{c} \xrightarrow{? \otimes_A^L M} \\ \xleftarrow{\text{RHom}_B(M, ?)} \end{array} \mathcal{D}(B).$$

- Let $f : A \rightarrow B$ be a quasi-isomorphism of dg algebras. Then the induced triangle functor $? \otimes_A^L B : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is an equivalence. A quasi-inverse is given by the restriction $\mathcal{D}(B) \rightarrow \mathcal{D}(A)$ along f . It can be written as $? \otimes_B^L B = \text{RHom}_B(B, ?)$ where B is considered as a dg B - A -bimodule respectively dg A - B -bimodule via f . These equivalences restrict to equivalences between $\text{per}(A)$ and $\text{per}(B)$ and, if k is a field, between $\mathcal{D}_{fd}(A)$ and $\mathcal{D}_{fd}(B)$. By abuse of language, by a quasi-isomorphism we will also mean a zigzag of quasi-isomorphisms.

2.3. The Nakayama functor. Let k be a field and let A be a dg k -algebra. We consider the dg functor $\nu = D\mathcal{H}om_A(? , A) : \mathcal{C}(A) \rightarrow \mathcal{C}(A)$. It is clear that $\nu(A) = D(A)$ holds. Moreover, for dg A -modules M and N there is a binatural map

$$\begin{aligned} D\mathcal{H}om_A(M, N) &\longrightarrow \mathcal{H}om_A(N, \nu(M)) \\ \varphi &\longmapsto (n \mapsto (f \mapsto \varphi(g))) \end{aligned} \tag{2.1}$$

where $f \in \mathcal{H}om_A(M, A)$ and $g : m \mapsto nf(m)$. If we let $M = A$, then (2.1) is an isomorphism and hence a quasi-isomorphism for $M \in \text{per}(A)$. By abuse of notation the left derived functor of ν is also denoted by ν . Passing to the derived category in (2.1) yields a binatural isomorphism for $M \in \text{per}(A)$ and $N \in \mathcal{D}(A)$:

$$D\text{Hom}_{\mathcal{D}(A)}(M, N) \cong \text{Hom}_{\mathcal{D}(A)}(N, \nu(M)). \tag{2.2}$$

One recovers the *Auslander–Reiten formula* if A is a finite-dimensional k -algebra.

2.4. Non-positive dg algebras: t -structures and co- t -structures. Let \mathcal{C} be a triangulated k -category with shift functor $[1]$. A t -structure on \mathcal{C} ([15]) is a pair $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of strictly (i.e. closed under isomorphisms) full subcategories such that

- $\mathcal{C}^{\leq 0}[1] \subseteq \mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}[-1] \subseteq \mathcal{C}^{\geq 0}$,
- $\mathrm{Hom}(M, N[-1]) = 0$ for $M \in \mathcal{C}^{\leq 0}$ and $N \in \mathcal{C}^{\geq 0}$,
- for each $M \in \mathcal{C}$ there is a triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ in \mathcal{C} with $M' \in \mathcal{C}^{\leq 0}$ and $M'' \in \mathcal{C}^{\geq 0}[-1]$.

The heart $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ of the t -structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ is an abelian category ([15]).

Let A be a dg k -algebra such that $A^i = 0$ for $i > 0$. Such a dg algebra is called a *non-positive dg algebra*. The canonical projection $A \rightarrow H^0(A)$ is a homomorphism of dg algebras. We view a module over $H^0(A)$ as a dg module over A via this homomorphism. This defines a natural functor $\mathrm{Mod} - H^0(A) \rightarrow \mathcal{D}(A)$.

Proposition 2.5. *Let A be a non-positive dg algebra.*

- (a) ([42, Theorem 1.3], [2, Section 2.1] and [56, Section 5.1]) *Let $\mathcal{D}^{\leq 0}$ respectively $\mathcal{D}^{\geq 0}$ denote the full subcategory of $\mathcal{D}(A)$ which consists of objects M such that $H^i(M) = 0$ for $i > 0$ respectively for $i < 0$. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t -structure on $\mathcal{D}(A)$. Moreover, taking H^0 is an equivalence from the heart to $\mathrm{Mod} - H^0(A)$, and the natural functor $\mathrm{Mod} - H^0(A) \rightarrow \mathcal{D}(A)$ induces a quasi-inverse to this equivalence. We will identify $\mathrm{Mod} - H^0(A)$ with the heart via these equivalences.*
- (b) *Assume that k is a field. The t -structure in (a) restricts to a t -structure on $\mathcal{D}_{fd}(A)$ whose heart is $\mathrm{fdmod} - H^0(A)$. Moreover, as a triangulated category $\mathcal{D}_{fd}(A)$ is generated by the heart.*
- (c) *Assume that k is a field. Assume that $\mathcal{D}_{fd}(A) \subseteq \mathrm{per}(A)$ and $\mathrm{per}(A)$ is Hom-finite. Then the t -structure in (a) restricts to a t -structure on $\mathrm{per}(A)$, whose heart is $\mathrm{fdmod} - H^0(A)$.*

Proof. (a) We only give the key point. Let M be a dg A -module. Thanks to the assumption that A is non-positive, the standard truncations $\sigma^{\leq 0}M$ and $\sigma^{> 0}M$ are again dg A -modules. A dg A -module M whose cohomologies are concentrated in degree 0 is related to the $H^0(A)$ -module $H^0(M)$ via the following chain of quasi-isomorphisms $H^0(M) \leftarrow \sigma^{\leq 0}M \hookrightarrow M$. Moreover, we have a distinguished triangle

$$\sigma^{\leq 0}M \longrightarrow M \longrightarrow \sigma^{> 0}M \longrightarrow \sigma^{\leq 0}M[1] \quad (2.3)$$

in $\mathcal{D}(A)$. This is the triangle required to show that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t -structure.

(b) For the first statement, it suffices to show that, under the assumptions, the standard truncations are endo-functors of $\mathcal{D}_{fd}(A)$. This is true because $H^*(\sigma^{\leq 0}M)$ and $H^*(\sigma^{> 0}M)$ are subspaces of $H^*(M)$.

To show the second statement, let $M \in \mathcal{D}_{fd}(M)$. Suppose that for $m \geq n$ we have $H^n(M) \neq 0$, $H^m(M) \neq 0$ but $H^i(M) = 0$ for $i > m$ or $i < n$. We prove that M is generated by the heart by induction on $m - n$. If $m - n = 0$, then a shift of M is in

the heart. Now suppose $m - n > 0$. The standard truncations yield a triangle

$$\sigma^{\leq n} M \longrightarrow M \longrightarrow \sigma^{> n} M \longrightarrow \sigma^{\leq n} M[1].$$

Now the cohomologies of $\sigma^{\leq n} M$ are concentrated in degree n , and hence $\sigma^{\leq n} M$ belongs to a shifted copy of the heart. Moreover, the cohomologies of $\sigma^{> n}(M)$ are bounded between degrees $n+1$ and m . By induction hypothesis $\sigma^{> n}(M)$ is generated by the heart. Therefore M is generated by the heart.

(c) Same as the proof for [2, Proposition 2.7]. \square

Let \mathcal{C} be as above. A *co-t-structure* on \mathcal{C} [74] (or *weight structure* [16]) is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of strictly full subcategories of \mathcal{C} satisfying the following conditions

- both $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are closed under finite direct sums and direct summands,
- $\mathcal{C}_{\geq 0}[-1] \subseteq \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}[1] \subseteq \mathcal{C}_{\leq 0}$,
- $\text{Hom}(M, N[1]) = 0$ for $M \in \mathcal{C}_{\geq 0}$ and $N \in \mathcal{C}_{\leq 0}$,
- for each $M \in \mathcal{C}$ there is a triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ in \mathcal{C} with $M' \in \mathcal{C}_{\geq 0}$ and $M'' \in \mathcal{C}_{\leq 0}[1]$.

It follows that $\mathcal{C}_{\leq 0} = \mathcal{C}_{\geq 0}^{\perp}[-1]$. The *co-heart* is defined as the intersection $\mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$.

Lemma 2.6. ([16, Proposition 1.3.3.6]) *For $M \in \mathcal{C}_{\leq 0}$, there exists a distinguished triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ with $M' \in \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ and $M'' \in \mathcal{C}_{\leq 0}[1]$.*

Let A be a non-positive dg k -algebra. Let $\mathcal{P}_{\geq 0}$ respectively $\mathcal{P}_{\leq 0}$ denote the smallest full subcategory of $\text{per}(A)$ which contains $A[i]$ for $i \leq 0$ respectively $i \geq 0$ and is closed under taking extensions and direct summands. We need a result from [16].

Proposition 2.7. *$(\mathcal{P}_{\geq 0}, \mathcal{P}_{\leq 0})$ is a co-t-structure of $\text{per}(A)$, with co-heart $\text{add}(A)$.*

Proof. This follows from [16, Proposition 5.2.2, Proposition 6.2.1]. \square

Hence, objects in $\mathcal{P}_{\leq 0}$ are characterised by the vanishing of the positive cohomologies:

Corollary 2.8. $\mathcal{P}_{\leq 0} = \{M \in \text{per}(A) \mid H^i(M) = 0 \text{ for any } i > 0\}$.

Proof. Let \mathcal{S} be the category on the right. By the preceding proposition, $\mathcal{P}_{\leq 0} = \mathcal{P}_{\geq 0}^{\perp}[-1] = (\mathcal{P}_{\geq 0}[-1])^{\perp}$. In particular, for $M \in \mathcal{P}_{\leq 0}$ and $i < 0$ this implies that $\text{Hom}(A[i], M) = 0$. Hence, $H^i(M) = 0$ holds for any $i > 0$ and M is in \mathcal{S} . Conversely, if $H^i(M) = \text{Hom}(A[-i], M) = 0$ for any $i > 0$, then it follows by dévissage that $\text{Hom}(N, M) = 0$ for any $N \in \mathcal{P}_{\geq 0}[-1]$. This shows that M is contained in $\mathcal{P}_{\leq 0}$. \square

2.9. Non-positive dg algebras: Hom-finiteness. Let A be a dg algebra. Then the subcomplex $\sigma^{\leq 0} A$ inherits a dg algebra structure from A . Therefore if $H^i(A) = 0$ for any $i > 0$, the embedding $\sigma^{\leq 0} A \hookrightarrow A$ is a quasi-isomorphism of dg algebras.

We generalise [2, Lemma 2.5 & Prop. 2.4] and [40, Lemma 2.4 & Prop. 2.5].

Proposition 2.10. *Let k be a field and A be a dg k -algebra such that*

- $A^i = 0$ for any $i > 0$,
- $H^0(A)$ is finite-dimensional,

$$- \mathcal{D}_{fd}(A) \subseteq \text{per}(A).$$

Then $H^i(A)$ is finite-dimensional for any i . Consequently, $\text{per}(A)$ is Hom-finite.

Proof. It suffices to prove the following induction step: if $H^i(A)$ is finite-dimensional for $-n \leq i \leq 0$, then $H^{-n-1}(A)$ is finite-dimensional.

To prove this claim, we consider the triangle induced by the standard truncations

$$\sigma^{\leq -n-1}A \longrightarrow A \longrightarrow \sigma^{> -n-1}A \longrightarrow (\sigma^{\leq -n-1}A)[1].$$

Since $H^i(\sigma^{> -n-1}A) = H^i(A)$ for $i \geq -n$, it follows by the induction hypothesis that $\sigma^{> -n-1}A$ belongs to $\mathcal{D}_{fd}(A)$, and hence to $\text{per}(A)$ by the third assumption on A . Therefore $\sigma^{\leq -n-1}A \in \text{per}(A)$. By Corollary 2.8, $(\sigma^{\leq -n-1}A)[-n-1] \in \mathcal{P}_{\leq 0}$. Moreover, Lemma 2.6 and Proposition 2.7 imply that there is a triangle

$$M' \longrightarrow (\sigma^{\leq -n-1}A)[-n-1] \longrightarrow M'' \longrightarrow M'[1]$$

with $M' \in \text{add}(A)$ and $M'' \in \mathcal{P}_{\leq 0}[1]$. It follows from Corollary 2.8 that $H^0(M'') = 0$. Therefore applying H^0 to the above triangle, we obtain an exact sequence

$$H^0(M') \longrightarrow H^0((\sigma^{\leq -n-1}A)[-n-1]) = H^{-n-1}(A) \longrightarrow 0.$$

But $H^0(M')$ is finite-dimensional because $M' \in \text{add}(A)$ holds and $H^0(A)$ has finite dimension by assumption. Thus $H^{-n-1}(A)$ is finite-dimensional. \square

2.11. Minimal relations. Let k be a field and Q be a finite quiver. Denote by \widehat{kQ} the *complete path algebra* of Q , i.e. the completion of the path algebra kQ with respect to the \mathfrak{m} -adic topology, where \mathfrak{m} is the ideal of kQ generated by all arrows. Namely, \widehat{kQ} is the inverse limit in the category of algebras of the inverse system $\{kQ/\mathfrak{m}^n, \pi_n : kQ/\mathfrak{m}^{n+1} \rightarrow kQ/\mathfrak{m}^n\}_{n \in \mathbb{N}}$, where π_n is the canonical projection. Later we will also work with complete path algebras of graded quivers: they are defined as above with the inverse limit taken in the category of graded algebras.

The complete path algebra \widehat{kQ} has a natural topology, the J -adic topology for J the ideal generated by all arrows. Let I be a closed ideal of \widehat{kQ} contained in J^2 and let $A = \widehat{kQ}/I$. For a vertex i of Q , let e_i denote the trivial path at i . A *set of minimal relations* of A (or of I) is a finite subset R of $\bigcup_{i,j \in Q_0} e_i I e_j$ such that I coincides with the closure $\overline{(R)}$ of the ideal of \widehat{kQ} generated by R but not with $\overline{(R')}$ for any proper subset R' of R . The following result, which is known to the experts, generalises [17, Proposition 1.2] (cf. also [53, Section 6.9]).

Proposition 2.12. *Let i and j be vertices of Q . If $e_i R e_j = \{r_1, \dots, r_s\}$, then the equivalence classes $\bar{r}_1, \dots, \bar{r}_s$ form a basis of $e_i(I/(IJ + JI))e_j$. In particular, the cardinality of $e_i R e_j$ does not depend on the choice of R .*

Proof. We show that $\bar{r}_1, \dots, \bar{r}_s$ are linearly independent. Otherwise, there would be elements $\lambda_1, \dots, \lambda_s$ in k and a relation $\sum_{a=1}^s \lambda_a r_a = 0 \pmod{IJ + JI}$, where without loss of generality $\lambda_1 \neq 0$. In other words, there exists an index set Γ and elements

$c_\gamma \in e_i \widehat{kQ}$ and $c^\gamma \in \widehat{kQ} e_j$ such that for any fixed $\gamma \in \Gamma$ at least one of c_γ and c^γ belongs to J and such that $\sum_{a=1}^s \lambda_a r_a = \sum_{r \in R} \sum_{\gamma \in \Gamma} c_\gamma r c^\gamma$ holds. Then we have

$$r_1 = -\lambda_1^{-1} \sum_{a=2}^s \lambda_a r_a + \lambda_1^{-1} \sum_{r \in R} \sum_{\gamma \in \Gamma} c_\gamma r c^\gamma.$$

Writing the right hand side as $f(r_1)$ and proceeding iteratively, we see that $r_1 = \lim_n f^n(r_1)$ holds. Hence, $r_1 \in \overline{(R \setminus \{r_1\})}$, contradicting the minimality of R . \square

For non-complete presentations of algebras, this result fails in general, see for example [75, Example 4.3].

2.13. The dual bar construction. In this subsection we recall definition and basic properties of the dual bar construction of an A_∞ -algebra. Our main references are [62, 63, 61]. Notice that [62, Lemma 11.1, Theorem 11.2] do not apply to our setting to obtain Theorem 2.14 and the interpretation of the Koszul dual in the derived category, but their proofs can be adapted.

Let k be a field. An A_∞ -algebra A is a graded k -vector space endowed with a family of homogenous k -linear maps of degree 1 (called *multiplications*) $\{b_n : (A[1])^{\otimes n} \rightarrow A[1] | n \geq 1\}$ satisfying the following identities

$$\sum_{j+k+l=n} b_{j+1+l}(id^{\otimes j} \otimes b_k \otimes id^{\otimes l}) = 0, \quad n \geq 1. \quad (2.4)$$

For example, a dg algebra can be viewed as an A_∞ -algebra with vanishing b_n for $n \geq 3$. A is said to be *minimal* if $b_1 = 0$. Now, suppose that either A satisfies

- $A^i = 0$ for all $i < 0$,
- A^0 is the product of r copies of the base field k for some positive integer r ,
- $b_n(a_1 \otimes \cdots \otimes a_n) = 0$ if one of a_1, \dots, a_n belongs to A^0 and $n \neq 2$.

or A satisfies

- $A^i = 0$ for all $i > 0$,
- $H^0(A) \cong \widehat{kQ}/\overline{(R)}$, for a finite quiver Q and a set R of minimal relations,
- $b_n(a_1 \otimes \cdots \otimes a_n) = 0$ if one of a_1, \dots, a_n is the trivial path at some vertex and $n \neq 2$.

Let $K = A^0$ in the former case and $K = H^0(A)/\text{rad } H^0(A)$ in the latter case. In both cases, there is an injective homomorphism $\eta : K \rightarrow A$ and surjective homomorphism $\varepsilon : A \rightarrow K$ of A_∞ -algebras. Denote by $\bar{A} = \ker \varepsilon$. Note that \bar{A} inherits the structure of an A_∞ -algebra. The *bar construction* of A , denoted by BA , is the graded vector space

$$T_K(\bar{A}[1]) = K \oplus \bar{A}[1] \oplus \bar{A}[1] \otimes_K \bar{A}[1] \oplus \dots$$

It is naturally a coalgebra with comultiplication defined by splitting the tensors. Moreover, $\{b_n | n \geq 1\}$ uniquely extends to a differential on BA which makes it a dg coalgebra. The *Koszul dual* of A is the graded k -dual of BA :

$$E(A) = B^\# A := D(BA).$$

It is a dg algebra and admits a *pseudo-compact* topology which is compatible with the dg-algebra structure (so it is a *pseudo-compact dg algebra*, see [56, 89]). As a graded algebra $E(A) = \widehat{T}_K(D(\bar{A}[1]))$ is the complete tensor algebra of $D(\bar{A}[1]) = \mathbf{Hom}_k(\bar{A}[1], k)$ over K . Its differential d is the unique continuous k -linear map satisfying the graded Leibniz rule and taking $f \in D(\bar{A}[1])$ to $d(f) \in B^\#A$, defined by

$$d(f)(a_1 \otimes \cdots \otimes a_n) = f(b_n(a_1 \otimes \cdots \otimes a_n)), \quad a_1, \dots, a_n \in \bar{A}[1].$$

Let \mathfrak{m} be the ideal of $E(A)$ generated by $D(\bar{A}[1])$. Then A being minimal amounts to say that $d(\mathfrak{m}) \subseteq \mathfrak{m}^2$ holds true.

If A and B are A_∞ -isomorphic, then $E(A)$ and $E(B)$ are quasi-isomorphic. Inside the derived category, $E(A)$ can be interpreted as the dg endomorphism algebra $\mathbf{RHom}_{\mathcal{D}(A)}(K, K)$, where K is viewed as an A_∞ -module via the homomorphism ε , see [63]. In particular, $H^*(E(A))$ is isomorphic to $\bigoplus_{i \in \mathbb{Z}} \mathbf{Hom}(K, K[i])$. The minimal model [48] of $E(A)$ is called the A_∞ -Koszul dual of A and is denoted by A^* .

Theorem 2.14. ([63, Theorem 2.4]) *Let A be an A_∞ -algebra as above. If the space $H^i(A)$ is finite-dimensional for each $i \in \mathbb{Z}$, then $E(E(A))$ is A_∞ -isomorphic to A . In particular, A is A_∞ -isomorphic to $E(A^*)$.*

If A is a dg algebra, then the A_∞ -isomorphism in the theorem can be replaced by a quasi-isomorphism of dg algebras, see [64, Proposition 2.8].

Moreover, we can describe $E(A)$ in terms of quivers. For $m \in \mathbb{Z}$, let Q_1^m be a k -basis of the degree m component of $D(\bar{A}[1])$ such that each basis element spans an $A^0 \otimes_k A^0$ -submodule. Recall that A^0 is a product of r copies of k . Let e_1, \dots, e_r be the standard basis of A^0 . Define Q as the graded quiver whose vertices are $1, \dots, r$ and whose set of arrows from i to j of degree m is $e_j Q_1^m e_i$. Then as a graded algebra $E(A)$ is the complete path algebra \widehat{kQ} of the graded quiver Q .

2.15. Recollements generated by idempotents. In this subsection our object of study is the triangle quotient $K^b(\mathbf{proj} - A) / \mathbf{thick}(eA)$, where A is an algebra and $e \in A$ is an idempotent. By Keller's Morita theorem for triangulated categories [52, Theorem 3.8 b)], the idempotent completion of this category is equivalent to the perfect derived category $\mathbf{per}(B)$ of some dg algebra B . Below we show that we can choose B such that there is a homomorphism of dg algebras $A \rightarrow B$, the restriction $\mathcal{D}(B) \rightarrow \mathcal{D}(A)$ along which is fully faithful.

Following [15], a *recollement* of triangulated categories is a diagram

$$\begin{array}{ccccc} & i^* & & j_! & \\ \mathcal{T}'' & \xleftarrow{\quad} & \mathcal{T} & \xleftarrow{\quad} & \mathcal{T}' \\ & i_* = i_! & & j^! = j^* & \\ & i^! & & j_* & \end{array} \quad (2.5)$$

of triangulated categories and triangle functors such that

- 1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples;
- 2) $j_!, i_* = i_!, j_*$ are fully faithful;

- 3) $j^*i_* = 0$;
 4) for every object X of \mathcal{T} there exist two distinguished triangles

$$i_!i^!X \rightarrow X \rightarrow j_*j^*X \rightarrow i_!i^!X[1] \quad \text{and} \quad j_!j^!X \rightarrow X \rightarrow i_*i^*X \rightarrow j_!j^!X[1],$$

where the morphisms starting from and ending at X are the units and counits.

In particular, $(\text{im } j_!, \text{im } i_*)$ and $(\text{im } i_*, \text{im } j_*)$ are two t -structures of \mathcal{T} . The triple $(\text{im } j_!, \text{im } i_*, \text{im } j_*)$ is a TTF triple, see [69, Section 2.1].

Let k be a commutative ring and A be a k -algebra. The following Proposition shows that every idempotent $e \in A$ gives rise to a recollement. In the literature, much attention has been paid to the special case that B has cohomologies concentrated in degree 0, see for example [28, 29, 27, 59]. Recall that A can be viewed as a dg k -algebra concentrated in degree 0 and in this case $\mathcal{D}(A) = \mathcal{D}(\text{Mod } -A)$.

Proposition 2.16. *Let A be a flat k -algebra and $e \in A$ an idempotent. There is a dg k -algebra B with a homomorphism of dg k -algebras $f: A \rightarrow B$ and a recollement of derived categories*

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}(B) & \xrightarrow{i_* = i_!} & \mathcal{D}(A) & \xrightarrow{j^! = j^*} & \mathcal{D}(eAe) \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array} \quad (2.6)$$

such that the following conditions are satisfied

- (a) the adjoint triples $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are given by

$$\begin{aligned} i^* &= ? \otimes_A^L B, & j_! &= ? \otimes_{eAe}^L eA, \\ i_* &= \text{RHom}_B(B, ?), & j^! &= \text{RHom}_A(eA, ?), \\ i_! &= ? \otimes_B^L B, & j^* &= ? \otimes_A^L Ae, \\ i^! &= \text{RHom}_A(B, ?), & j_* &= \text{RHom}_{eAe}(Ae, ?), \end{aligned}$$

where B is considered as an A - A -bimodule via the morphism f ;

- (b) $B^i = 0$ for $i > 0$;
 (c) $H^0(B)$ is isomorphic to A/AeA .

Remark 2.17. This result is known to hold in greater generality, see [34, Section 2 and 3] (which uses different terminologies). For convenience, we include a proof.

Proof. By the adjunction formula, the exact functor $\text{Hom}_A(eA, ?) = ? \otimes_A Ae: \text{Mod } -A \rightarrow \text{Mod } -eAe$ has both a left adjoint $? \otimes_{eAe} eA$ and a right adjoint $\text{Hom}_{eAe}(Ae, ?)$. Deriving these functors, we obtain the right half of the recollement (2.6). The derived functors are still adjoint and it is known that $? \otimes_{eAe}^L eA$ is fully faithful (see e.g. [51, Lemma 4.2]). An application of the Yoneda Lemma shows that there is a natural isomorphism $j^!j_! \cong 1_{\mathcal{D}(eAe)}$. Hence, $j^!$ is a quotient functor with left and right adjoints. In particular, it is a so called Bousfield localization and colocalization functor. Its kernel is $\mathcal{D}_{A/AeA}(A) \subseteq \mathcal{D}(A)$, the full subcategory of complexes with cohomologies in $\text{Mod } -A/AeA$. Hence, $j^!$ yields a recollement (see e.g. [68, Section 9])

$$\mathcal{D}_{A/AeA}(A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{D}(A) \begin{array}{c} \xleftarrow{j^!} \\ \xrightarrow{j^! = j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{D}(eAe) . \quad (2.7)$$

By [69, Theorem 4] (which needs the flatness assumption) and the first paragraph after Lemma 4 of loc. cit., there exists a dg algebra B' and a morphism of dg-algebras $f' : A \rightarrow B'$ such that there is a recollement (2.6) and the adjoint triple $(i^*, i_* = i_!, i^!)$ is given as in (a) with B replaced by B' . We claim that $H^i(B') = 0$ for $i > 0$ and that $H^0(f')$ induces an isomorphism of algebras $A/AeA \cong H^0(B')$. Then taking $B = \sigma^{\leq 0} B'$ and $f = \sigma^{\leq 0} f'$ finishes the proof for (a) (b) and (c).

In order to prove the claim, we take the distinguished triangle associated to A .

$$\begin{array}{ccccc} Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA & \xrightarrow{\varphi} & A & \xrightarrow{f'} & B' & \longrightarrow & Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA[1] \\ \parallel & & & & \parallel & & \\ j_! j^!(A) & & & & i_* i^*(A) & & \end{array} \quad (2.8)$$

By applying H^0 to the triangle (2.8) we obtain a long exact cohomology sequence

$$\cdots \cdots H^i(Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA) \xrightarrow{H^i(\varphi)} H^i(A) \xrightarrow{H^i(f')} H^i(B') \longrightarrow H^{i+1}(Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA) \cdots \cdots$$

If $i > 0$, both $H^i(A)$ and $H^{i+1}(Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA)$ are trivial, and hence $H^i(B')$ is trivial. If $i = 0$, then $H^0(B') \cong H^0(A)/\text{im}(H^0(\varphi))$. But $H^0(Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA) \cong Ae \otimes_{eAe} eA$ and the image of $H^0(\varphi)$ is precisely AeA . Therefore $H^0(f') : A \rightarrow H^0(B')$ induces an isomorphism $H^0(B') \cong A/AeA$, which is clearly a homomorphism of algebras. \square

Corollary 2.18. *Keep the assumptions and notations as in Proposition 2.16.*

(a) *The functor i^* induces an equivalence of triangulated categories*

$$(K^b(\text{proj } -A)/\text{thick}(eA))^\omega \xrightarrow{\sim} \text{per}(B), \quad (2.9)$$

where $(-)^\omega$ denotes the idempotent completion (see [14]).

(b) *Let k be a field. Let $\mathcal{D}_{fd,A/AeA}(A)$ be the full subcategory of $\mathcal{D}_{fd}(A)$ consisting of complexes with cohomologies supported on A/AeA . The functor i_* induces a triangle equivalence $\mathcal{D}_{fd}(B) \xrightarrow{\sim} \mathcal{D}_{fd,A/AeA}(A)$. Moreover, the latter category coincides with $\text{thick}_{\mathcal{D}(A)}(\text{fdmod } -A/AeA)$.*

Proof. (a) Since $j_!(eAe) = Ae \overset{\mathbf{L}}{\otimes}_{eAe} eA \cong eA$, eAe generates $\mathcal{D}(eAe)$ and $j_!$ commutes with direct sums, we obtain $\text{im } j_! = \text{Tria}(eA)$. Hence, i^* induces a triangle equivalence

$$\mathcal{D}(A)/\text{Tria}(eA) \cong \mathcal{D}(B). \quad (2.10)$$

As a projective A -module eA is compact in $\mathcal{D}(A)$. By definition, $\text{Tria}(eA)$ is the smallest localizing subcategory containing eA . Since $\mathcal{D}(A)$ is compactly generated, Neeman's interpretation (and generalization) [67, Theorem 2.1] of Thomason &

Trobaugh's and Yao's Localization Theorems shows that restricting (2.10) to the subcategories of compact objects yields a triangle equivalence $K^b(\mathbf{proj} - A)/\mathbf{thick}(eA) \rightarrow \mathbf{per}(B)$ up to direct summands. Hence, the equivalence (2.9) follows.

(b) By construction of the dg algebra B in Proposition 2.16, i_* induces a triangulated equivalence between $\mathcal{D}(B)$ and $\mathcal{D}_{A/AeA}(A)$, the full subcategory of $\mathcal{D}(A)$ consisting of complexes of A -modules which have cohomologies supported on A/AeA . Moreover, i_* restricts to a triangle equivalence between $\mathcal{D}_{fd}(B)$ and $i_*(\mathcal{D}_{fd}(B))$. The latter category is contained in $\mathcal{D}_{fd}(A)$ because i_* is the restriction along the homomorphism $f : A \rightarrow B$, and hence is contained in $\mathcal{D}_{fd}(A) \cap \mathcal{D}_{A/AeA}(A) = \mathcal{D}_{fd,A/AeA}(A)$, which in turn is contained in $\mathbf{thick}_{\mathcal{D}(A)}(\mathbf{fdmod} - A/AeA)$. By Proposition 2.5 (b), $\mathbf{fdmod} - H^0(B)$ generates $\mathcal{D}_{fd}(B)$. But i_* induces an equivalence from $\mathbf{fdmod} - H^0(B)$ to $\mathbf{fdmod} - A/AeA$. Therefore $\mathbf{thick}_{\mathcal{D}(A)}(\mathbf{fdmod} - A/AeA) = i_*(\mathcal{D}_{fd}(B))$, and hence $\mathbf{thick}_{\mathcal{D}(A)}(\mathbf{fdmod} - A/AeA) = i_*(\mathcal{D}_{fd}(B)) = \mathcal{D}_{fd,A/AeA}(A)$. We are done. \square

Remark 2.19. The triangle equivalences (2.9) and $\mathcal{D}(B) \cong \mathcal{D}_{A/AeA}(A)$ show that

$$(\mathcal{D}_{A/AeA}(A))^c \cong (K^b(\mathbf{proj} - A)/\mathbf{thick}(eA))^\omega. \quad (2.11)$$

In particular, if A is a non-commutative resolution of a complete Gorenstein singularity R , then the relative singularity category $\Delta_R(A) \cong \mathcal{D}^b(\mathbf{mod} - A)/\mathbf{thick}(eA)$ is idempotent complete by [23, Section 3]. Hence, there is a triangle equivalence

$$(\mathcal{D}_{A/AeA}(A))^c \cong \Delta_R(A). \quad (2.12)$$

3. A TALE OF TWO IDEMPOTENTS

Definition 3.1. A triangulated functor $\mathbb{F} : \mathcal{C} \rightarrow \mathcal{D}$ is called *triangulated quotient functor* if the induced functor $\underline{\mathbb{F}} : \mathcal{C}/\ker \mathbb{F} \rightarrow \mathcal{D}$ is an equivalence of categories.

Lemma 3.2. Let $\mathbb{F} : \mathcal{C} \rightarrow \mathcal{D}$ be a triangulated quotient functor with kernel \mathcal{K} . Let $\mathcal{U} \subseteq \mathcal{C}$ be a full triangulated subcategory, let $q : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{U}$ be the quotient functor and $\mathcal{V} = \mathbf{thick}(\mathbb{F}(\mathcal{U}))$. Then \mathbb{F} induces an equivalence of triangulated categories.

$$\frac{(\mathcal{C}/\mathcal{U})}{\mathbf{thick}(q(\mathcal{K}))} \longrightarrow \frac{\mathcal{D}}{\mathcal{V}}.$$

Proof. \mathbb{F} induces a triangle functor $\overline{\mathbb{F}} : \mathcal{C}/\mathcal{U} \rightarrow \mathcal{D}/\mathcal{V}$. We have $\mathbf{thick}(q(\mathcal{K})) \subseteq \ker(\overline{\mathbb{F}})$. To show that $\overline{\mathbb{F}}$ is universal with this property, let $\mathbb{G} : \mathcal{C}/\mathcal{U} \rightarrow \mathcal{T}$ be a triangle functor with $\mathbf{thick}(q(\mathcal{K})) \subseteq \ker(\mathbb{G})$. We explain the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{K} & \xrightarrow{\quad} & \mathcal{C} & \xrightarrow{\quad \mathbb{F} \quad} & \mathcal{D} \\ \downarrow q & & \downarrow q & & \downarrow q' \\ \mathbf{thick}(q(\mathcal{K})) & \xrightarrow{\quad} & \mathcal{C}/\mathcal{U} & \xrightarrow{\quad \overline{\mathbb{F}} \quad} & \mathcal{D}/\mathcal{V} \end{array} \quad \begin{array}{c} \mathbb{G} \nearrow \\ \mathbb{F} \searrow \end{array} \quad \begin{array}{c} \mathbb{I}_1 \text{ (dashed)} \\ \mathbb{I}_2 \text{ (dashed)} \end{array}$$

\mathbb{I}_1 exists by the universal property of \mathbb{F} and \mathbb{I}_2 exists by the universal property of q' . Since $\mathbb{I}_2 \circ \overline{\mathbb{F}} \circ q = \mathbb{I}_1 \circ \mathbb{F} = \mathbb{G} \circ q$ the universal property of q implies $\mathbb{I}_2 \circ \overline{\mathbb{F}} = \mathbb{G}$.

To show uniqueness of \mathbb{I}_2 let $\mathbb{H}: \mathcal{D}/\mathcal{V} \rightarrow \mathcal{T}$ be a triangle functor such that $\mathbb{H} \circ \overline{\mathbb{F}} = \mathbb{G}$. Then $\mathbb{H} \circ q' \circ \mathbb{F} = \mathbb{G} \circ q$ and the universal property of \mathbb{F} imply $\mathbb{H} \circ q' = \mathbb{I}_1$. Since $\mathbb{H} \circ q' = \mathbb{I}_1 = \mathbb{I}_2 \circ q'$ the universal property of q' yields $\mathbb{I}_2 = \mathbb{H}$. \square

Proposition 3.3. *Let A be a right Noetherian ring and let $e, f \in A$ be idempotents. The exact functor $\mathbb{F} = \text{Hom}_A(eA, -)$ induces an equivalence of triangulated categories*

$$\frac{\mathcal{D}^b(\text{mod } -A)/\text{thick}(fA)}{\text{thick}(q(\text{mod } -A/AeA))} \longrightarrow \frac{\mathcal{D}^b(\text{mod } -eAe)}{\text{thick}(fAe)}. \quad (3.1)$$

Proof. On the abelian level \mathbb{F} induces a well-known equivalence

$$\underline{\mathbb{F}}: \frac{\text{mod } -A}{\text{mod } -A/AeA} \longrightarrow \text{mod } -eAe, \quad (3.2)$$

which may be deduced from an appropriate version of [35, Proposition III.5] in conjunction with classical Morita theory (see e.g. [33, Theorem 8.4.4.]). Using a compatibility result, which relates abelian quotients with triangulated quotients of derived categories [66, Theorem 3.2.], the equivalence (3.2) shows that \mathbb{F} induces a triangulated *quotient* functor $\mathbb{F}: \mathcal{D}^b(\text{mod } -A) \rightarrow \mathcal{D}^b(\text{mod } -eAe)$. An application of Lemma 3.2 to \mathbb{F} and $\text{thick}(fA)$ completes the proof. \square

Remark 3.4. Proposition 3.3 contains Chen's [25, Theorem 3.1] as a special case. Namely, if we set $f = 1$ and assume that $\text{pr. dim}_{eAe}(Ae) < \infty$ holds, (3.1) yields a triangle equivalence $\mathcal{D}_{sg}(A)/\text{thick}(q(\text{mod } -A/AeA)) \rightarrow \mathcal{D}_{sg}(eAe)$. If moreover every finitely generated A/AeA -module has finite projective dimension over A (i.e. the idempotent e is *singularly-complete* in the terminology of loc. cit.), we get an equivalence $\mathcal{D}_{sg}(A) \rightarrow \mathcal{D}_{sg}(eAe)$ of singularity categories [25, Corollary 3.3].

Remark 3.5. Proposition 3.3 has an analogue for (non-commutative) ringed spaces $\mathbb{X} = (X, \mathcal{A})$ as studied in [23]. Let $j: U \rightarrow X$ be an open immersion. The restriction functor $j^*: \text{Perf}(X) \rightarrow \text{Perf}(U)$ is essentially surjective by [86, Lemma 5.5.1]. Moreover, $j^*: \mathcal{D}^b(\text{Coh}(\mathcal{A})) \rightarrow \mathcal{D}^b(\text{Coh}(\mathcal{A}_U))$ is a triangulated quotient functor. Hence, Lemma 3.2 yields an equivalence of triangulated categories

$$\frac{\mathcal{D}^b(\text{Coh}(\mathcal{A}))/\text{Perf}(X)}{\text{thick}(q(\text{Coh}_{(X \setminus U)}(\mathcal{A})))} \longrightarrow \frac{\mathcal{D}^b(\text{Coh}(\mathcal{A}_U))}{\text{Perf}(U)} \quad (3.3)$$

In combination with [23, Proposition 2.6] this yields a proof of [23, Theorem 2.7]. This is analogous to the commutative case treated in [25, Proposition 1.2].

4. FROM CLASSICAL TO RELATIVE SINGULARITY CATEGORIES

Let k be an algebraically closed field. In this section we study the *relative Auslander singularity category* $\Delta_{\mathcal{E}}(A)$ (4.9) of a Frobenius category \mathcal{E} which has only finitely many isoclasses of indecomposable objects and which satisfies certain additional conditions, where A is the Auslander algebra of \mathcal{E} . We describe the dg model of $\Delta_{\mathcal{E}}(A)$ in terms of the Auslander–Reiten quiver of the stable category $\underline{\mathcal{E}}$ of \mathcal{E} . This

is based on a study of the fractional Calabi–Yau property of simple A -modules which are not supported on the projective-injective generator of \mathcal{E} .

4.1. The fractional Calabi–Yau property. Let \mathcal{E} be an idempotent complete Frobenius k -category.

Definition 4.2. Let \mathcal{C} be an additive subcategory of \mathcal{E} . We say that \mathcal{C} *has d -almost split sequences* if \mathcal{C} is Krull–Schmidt and for any non-projective indecomposable object X of \mathcal{C} (respectively, non-injective indecomposable object Y of \mathcal{C}) there is an exact sequence (called a *d -almost split sequence*, see [45, Section 3.1])

$$0 \longrightarrow Y \xrightarrow{f_d} C_{d-1} \xrightarrow{f_{d-1}} \dots \longrightarrow C_0 \xrightarrow{f_0} X \longrightarrow 0$$

with terms in \mathcal{C} and f_d, \dots, f_0 belong to the Jacobson radical $J_{\mathcal{C}}$ of \mathcal{C} such that the following two sequences of functors are exact

$$0 \longrightarrow (?, Y) \xrightarrow{f_d} (?, C_{d-1}) \longrightarrow \dots \longrightarrow (?, C_0) \xrightarrow{f_0} J_{\mathcal{C}}(?, X) \longrightarrow 0,$$

$$0 \longrightarrow (X, ?) \xrightarrow{f_0} (C_0, ?) \longrightarrow \dots \longrightarrow (C_{d-1}, ?) \xrightarrow{f_d} J_{\mathcal{C}}(Y, ?) \longrightarrow 0,$$

where $(X, Y) = \text{Hom}_{\mathcal{E}}(X, Y)$. Denote $\tau_d(X) = Y$ (respectively, $\tau_d^{-1}(Y) = X$).

Assume that $\text{proj } \mathcal{E}$ has an additive generator P . Let $F = P \oplus F'$ be an object of \mathcal{E} such that F' has no projective direct summands. Let $A = \text{End}_{\mathcal{E}}(F)$, $R = \text{End}_{\mathcal{E}}(P)$ and $e = id_P \in A$. Then A/AeA is the stable endomorphism algebra of F . Recall from Corollary 2.18 (b) that $\mathcal{D}_{fd, A/AeA}(A)$ denotes the full subcategory of $\mathcal{D}_{fd}(A)$ consisting of complexes of A -modules which have cohomologies supported on A/AeA and that $\mathcal{D}_{fd, A/AeA}(A)$ is generated by $\text{fdmod} - A/AeA$.

Assume that $\text{add}(F)$ has d -almost split sequences. In particular, it is a Krull–Schmidt category. Hence, we may assume that $F' = F_1 \oplus \dots \oplus F_r$ such that F_1, \dots, F_r are pairwise non-isomorphic and non-projective indecomposable objects. Assume further that A/AeA is finite-dimensional over k . Then $\mathcal{D}_{fd, A/AeA}(A)$ is generated by the simple A/AeA -modules S_1, \dots, S_r , corresponding to F_1, \dots, F_r , respectively.

Theorem 4.3. *Assume that $\text{add}(F)$ has d -almost split sequences and that A/AeA is finite dimensional over k . Then the following statements hold.*

- (a) *Any finite dimensional A/AeA -module has finite projective dimension over A .*
- (b) *The triangulated category $\mathcal{D}_{fd, A/AeA}(A)$ admits a Serre functor ν .*
- (c) *For $i = 1, \dots, r$, the simple A/AeA -module S_i is fractionally $\frac{(d+1)n_i}{n_i} - CY$, where n_i is the smallest positive integer such that $\tau_d^{n_i}(F_i) \cong F_i$ holds.*
- (d) *There exists a permutation π on the isomorphism classes of simple A/AeA -modules such that $D \text{Ext}_A^l(S, S') \cong \text{Ext}_A^{d+1-l}(S', \pi(S))$, holds for all $l \in \mathbb{Z}$.*

Proof. For $i = 1, \dots, r$, let $e_i = \mathbf{1}_{F_i}$ and consider it as an element in A . Then $1_A = e + e_1 + \dots + e_r$. Let S_i be the simple A -module corresponding to e_i . By

assumption there is an d -almost split sequence (see Definition 4.2)

$$\eta: 0 \longrightarrow F_{\pi(i)} \longrightarrow C_{d-1} \longrightarrow \dots \longrightarrow C_0 \longrightarrow F_i \longrightarrow 0 \quad (4.1)$$

where $C_{d-1}, \dots, C_0 \in \mathbf{add}(F)$ and π is the permutation on the set $\{1, \dots, r\}$ induced by $F_{\pi(i)} = \tau_d(F_i)$. Applying $\mathbf{Hom}_{\mathcal{E}}(F, ?)$ to η yields a A -projective resolution of S_i

$$0 \longrightarrow (F, F_{\pi(i)}) \longrightarrow (F, C_{d-1}) \longrightarrow \dots \longrightarrow (F, C_0) \longrightarrow (F, F_i) \longrightarrow S_i \longrightarrow 0, \quad (4.2)$$

In particular, this shows (a). Dually, we acquire an A -injective resolution of $S_{\pi(i)}$

$$0 \rightarrow S_{\pi(i)} \rightarrow D(F_{\pi(i)}, F) \rightarrow D(C_{d-1}, F) \rightarrow \dots \rightarrow D(C_0, F) \rightarrow D(F_i, F) \rightarrow 0, \quad (4.3)$$

by applying $D \mathbf{Hom}_{\mathcal{E}}(?, F)$ to η . Recall from Section 2.3 that there is a triangle functor $\nu: \mathbf{per}(A) \rightarrow \mathbf{thick}(DA)$. We deduce from the above long exact sequences that $\mathcal{D}_{fd, A/AeA}(A) = \mathbf{thick}(S_1, \dots, S_r) \subseteq \mathbf{per}(A) \cap \mathbf{thick}(DA)$ and that $\nu(S_i) = S_{\pi(i)}[d+1]$. It follows from the Auslander–Reiten formula (2.2) that the restriction of ν on $\mathcal{D}_{fd, A/AeA}(A)$ is a right Serre functor and hence fully faithful [78, Corollary I.1.2]. Since, as shown above, ν takes a set of generators of $\mathcal{D}_{fd, A/AeA}(A)$ to itself up to shift, it follows that ν restricts to an auto-equivalence of $\mathcal{D}_{fd, A/AeA}(A)$. In particular, ν is a Serre functor of $\mathcal{D}_{fd, A/AeA}(A)$. Moreover, if n denotes the number of elements in the π -orbit of i , then $\nu^n(S_i) \cong S_i[(d+1)n]$, i.e. S_i is fractionally Calabi–Yau of Calabi–Yau dimension $\frac{(d+1)n}{n}$. Finally, we have a chain of isomorphisms

$$\begin{aligned} D \mathbf{Ext}_A^l(S_i, S_j) &\cong D \mathbf{Hom}_A(S_i, S_j[l]) \cong \mathbf{Hom}_A(S_j, \nu(S_i)[-l]) \\ &\cong \mathbf{Hom}_A(S_j, S_{\pi(i)}[d+1-l]) \cong \mathbf{Ext}_A^{d+1-l}(S_j, S_{\pi(i)}), \end{aligned}$$

where $i, j = 1, \dots, r$ and l denotes an integer. This proves part (d). \square

4.4. Independence of the Frobenius model. Let \mathcal{T} be an idempotent complete Hom-finite algebraic triangulated category with only finitely many isomorphism classes of indecomposable objects, say M_1, \dots, M_r . Then \mathcal{T} has a Serre functor [1, Theorem 1.1] and thus has Auslander–Reiten triangles [78, Theorem I.2.4]. Let τ be the Auslander–Reiten translation. By abuse of notation, τ also denotes the induced permutation on $\{1, \dots, r\}$ defined by $M_{\tau(i)} = \tau M_i$.

The quiver of the Auslander algebra $\Lambda(\mathcal{T}) = \mathbf{End}_{\mathcal{T}}(\bigoplus_{i=1}^r M_i)$ of \mathcal{T} is the Gabriel quiver Γ of \mathcal{T} , in which we identify i with M_i . We assume that there exists a sequence of elements $\gamma = \{\gamma_1, \dots, \gamma_r\}$ in $\widehat{k\Gamma}$, satisfying the following conditions:

- (A1) for each vertex i the element γ_i is a (possibly infinite) combination of paths of Γ from i to $\tau^{-1}i$,
- (A2) γ_i is non-zero if and only if Γ has at least one arrow starting in i ,
- (A3) the non-zero γ_i 's form a set of minimal relations for $\Lambda(\mathcal{T})$ (see Section 2.11).

Definition 4.5. The dg Auslander algebra $\Lambda_{dg}(\mathcal{T}, \gamma)$ of \mathcal{T} with respect to γ is the dg algebra (\widehat{kQ}, d) , where Q is a graded quiver and $d: \widehat{kQ} \rightarrow \widehat{kQ}$ is a map such that

- (dgA1) Q is concentrated in degrees 0 and -1 ,
- (dgA2) the degree 0 part of Q is the same as the Gabriel quiver Γ of \mathcal{T} ,

- (dgA3) for each vertex i , there is precisely one arrow $\rho_i: i \rightarrow \tau^{-1}(i)$ of degree -1 ,
 (dgA4) d is the unique continuous k -linear map on \widehat{kQ} of degree 1 satisfying the graded Leibniz rule and taking ρ_i ($i \in Q_0$) to the relation γ_i .

In fact, the dg Auslander algebra does not depend on the choice of the sequence γ :

Proposition 4.6. *Let \mathcal{T} be as above, and let $\gamma = \{\gamma_1, \dots, \gamma_r\}$ and $\gamma' = \{\gamma'_1, \dots, \gamma'_r\}$ be sequences of elements of $\widehat{k\Gamma}$ satisfying the conditions (A1)–(A3). Then the dg Auslander algebras $\Lambda_{dg}(\mathcal{T}, \gamma)$ and $\Lambda_{dg}(\mathcal{T}, \gamma')$ are isomorphic as dg algebras.*

Proof. By the assumptions (A1)–(A3), there exist $c_i \in k^\times$ ($i = 1, \dots, r$), an index set P and $c_{pi}, c^{pi} \in \widehat{k\Gamma}$ ($(p, i) \in P \times \{1, \dots, r\}$), such that for each pair (p, i) , at least one of c_{pi} and c^{pi} belongs to the ideal of $\widehat{k\Gamma}$ generated by all arrows, and

$$\gamma'_i = c_i \gamma_i + \sum_{j=1}^r \sum_{p \in P} c_{pj} \gamma_j c^{pj}. \quad (4.4)$$

We define a continuous graded k -algebra homomorphism $\varphi: \Lambda_{dg}(\mathcal{T}, \gamma') \rightarrow \Lambda_{dg}(\mathcal{T}, \gamma)$ as follows: it is the identity on the degree 0 part and for arrows of degree -1 we set

$$\varphi(\rho'_i) = c_i \rho_i + \sum_{j=1}^r \sum_{p \in P} c_{pj} \rho_j c^{pj}. \quad (4.5)$$

Since $\gamma_i = d(\rho_i)$ and $\gamma'_i = d(\rho'_i)$, it follows from (4.4) and (4.5) that φ is a homomorphism of dg algebras. The equation (4.5), yields

$$\rho_i = c_i^{-1} \varphi(\rho'_i) - c_i^{-1} \sum_{j=1}^r \sum_{p \in P} c_{pj} \rho_j c^{pj}. \quad (4.6)$$

By iteratively substituting $c_j^{-1} \varphi(\rho'_j) - c_j^{-1} \sum_{k=1}^r \sum_{p \in P} c_{pk} \rho_k c^{pk}$ for ρ_j on the right hand side of (4.6), we see that there exists an index set P' and elements $c'_{pi}, c'^{pi} \in \widehat{k\Gamma}$ ($(p, i) \in P' \times \{1, \dots, r\}$) such that for each pair (p, i) at least one of c'_{pi} and c'^{pi} belongs to the ideal of $\widehat{k\Gamma}$ generated by all arrows, and the following equation holds

$$\rho_i = c_i^{-1} \varphi(\rho'_i) - \sum_{j=1}^r \sum_{p \in P'} c'_{pj} \varphi(\rho'_j) c'^{pj}. \quad (4.7)$$

Define a continuous graded k -algebra homomorphism $\varphi': \Lambda_{dg}(\mathcal{T}, \gamma) \rightarrow \Lambda_{dg}(\mathcal{T}, \gamma')$ as follows: φ' is the identity on the degree 0 part and for arrows of degree -1 we set:

$$\varphi'(\rho_i) = c_i^{-1} \rho'_i - \sum_{j=1}^r \sum_{p \in P'} c'_{pj} \rho'_j c'^{pj}. \quad (4.8)$$

It is clear that $\varphi \circ \varphi' = id$ holds. Since φ' and φ have a similar form, the same argument as above shows that there exists a continuous graded k -algebra homomorphism $\varphi'': \Lambda_{dg}(\mathcal{T}, \gamma') \rightarrow \Lambda_{dg}(\mathcal{T}, \gamma)$ such that $\varphi' \circ \varphi'' = id$ holds. Therefore we have $\varphi = \varphi''$. In particular, we see that φ is an isomorphism. \square

Henceforth, we denote by $\Lambda_{dg}(\mathcal{T})$ the dg Auslander algebra of \mathcal{T} with respect to any sequence γ satisfying (A1)–(A3). By definition of \mathcal{T} , there is a triangle equivalence $\mathcal{T} \cong \underline{\mathcal{E}}$, for a Frobenius category \mathcal{E} . We assume that \mathcal{E} additionally satisfies:

- (FM1) \mathcal{E} is idempotent complete and $\mathbf{proj} \mathcal{E}$ has an additive generator P ,
- (FM2) \mathcal{E} has only finitely many isoclasses of indecomposable objects, N_1, \dots, N_s ,
- (FM3) \mathcal{E} has (1-) almost split sequences,
- (FM4) the Auslander algebra $A = \mathbf{End}_{\mathcal{E}}(\bigoplus_{i=1}^s N_i)$ of \mathcal{E} is right Noetherian.

Let $e \in A$ be the idempotent endomorphism corresponding to $\mathbf{1}_P$, where P denotes the additive generator of $\mathbf{proj} \mathcal{E}$. In analogy with the special case $\mathcal{E} = \mathbf{MCM}(R)$, for a Gorenstein ring. We define the *relative Auslander singularity category* as follows

$$\Delta_{\mathcal{E}}(A) = \frac{K^b(\mathbf{proj} - A)}{\mathbf{thick}(eA)}. \quad (4.9)$$

If $\mathcal{E} = \mathbf{MCM}(R)$ for a Gorenstein ring R , then $\Delta_{\mathcal{E}}(A)$ is equivalent to the relative singularity category $\Delta_R(\mathbf{Aus}(R))$ as defined in the introduction (1.1).

Theorem 4.7. *Let \mathcal{E} be a Frobenius category satisfying conditions (FM1)–(FM4). If $\mathcal{T} := \underline{\mathcal{E}}$ is Hom-finite and idempotent complete, then the following statements hold*

- (a) *there is a sequence γ of minimal relations for the Auslander algebra of \mathcal{T} satisfying the above conditions (A1)–(A3),*
- (b) *$\Delta_{\mathcal{E}}(A)$ is triangle equivalent to $\mathbf{per}(\Lambda_{dg}(\mathcal{T}))$ (up to direct summands),*
- (c) *$\Delta_{\mathcal{E}}(A)$ is Hom-finite.*

Remark 4.8. If $\Delta_{\mathcal{E}}(A)$ is idempotent complete, then we can omit the supplement “up to direct summands” in the statement above. In particular, this holds in the case $\mathcal{E} = \mathbf{MCM}(R)$, where (R, \mathfrak{m}) is a local complete Gorenstein (R/\mathfrak{m}) -algebra [23].

Proof. By Corollary 2.18 (a), there exists a non-positive dg algebra B with $H^0(B) \cong A/AeA$, such that $(\Delta_{\mathcal{E}}(A))^{\omega}$ is triangle equivalent to $\mathbf{per}(B)$. Hence, it suffices to show that (a) holds, that B is quasi-isomorphic to $\Lambda_{dg}(\mathcal{T})$ and that $\mathbf{per}(B)$ is Hom-finite.

As shown in the proof of Theorem 4.3, all simple A/AeA -modules and hence all finite-dimensional A/AeA -modules have finite projective dimension over A . It follows from Proposition 2.10 that $H^i(B)$ is finite-dimensional over k for any $i \in \mathbb{Z}$ and $\mathbf{per}(B)$ is Hom-finite. So by Theorem 2.14, we have that B is quasi-isomorphic to $E(B^*)$, where B^* is the A_{∞} -Koszul dual of B . Let S_1, \dots, S_r be a complete set of non-isomorphic simple A/AeA -modules and let $S = \bigoplus_{i=1}^r S_i$. Then B^* is the minimal model of $\mathbf{RHom}_B(S, S) = \mathbf{RHom}_A(S, S)$. In particular, as a graded algebra, B^* is isomorphic to $\mathbf{Ext}_A^*(S, S)$. It follows from Theorem 4.3 that $\mathbf{Ext}_A^*(S, S)$ is concentrated in degrees 0, 1 and 2. Clearly $\mathbf{Ext}_A^0(S_i, S_j) = 0$ unless $i = j$ in which case it is k . A careful analysis of the proof of Theorem 4.3 tells us that in the current situation the permutation π coincides with τ . Therefore, $\mathbf{Ext}_A^2(S_i, S_j) = 0$ unless $j = \tau(i)$. Hence, $E(B^*) = (\widehat{kQ}, d)$ for a graded quiver Q and a continuous k -linear differential d of degree 1, where the graded quiver Q is concentrated in degree

0 and -1 , and starting from any vertex i there is precisely one arrow ρ_i of degree -1 whose target is $\tau^{-1}i$.

Let Q^0 denote the degree 0 part of Q . Then $H^0(E(B^*)) = \widehat{kQ^0}/\overline{d(\rho_i)}$. Since $H^0(E(B^*)) \cong H^0(B) \cong A/AeA = \Lambda(\mathcal{T})$ is the Auslander algebra of \mathcal{T} , it follows that Q^0 is the same as the Gabriel quiver Γ of \mathcal{T} . Moreover, $\gamma = \{d(\rho_1), \dots, d(\rho_r)\}$ is a set of relations for $\Lambda(\mathcal{T})$. We claim that γ is a sequence satisfying the conditions (A1)–(A3). Then (a) holds and $E(B^*) = \Lambda_{dg}(\mathcal{T}, \gamma) = \Lambda_{dg}(\mathcal{T})$, which implies that B is quasi-isomorphic to $\Lambda_{dg}(\mathcal{T})$.

Since we already know that $\rho_i: i \dashrightarrow \tau^{-1}(i)$ holds, $d(\rho_i)$ is a combination of paths from i to $\tau^{-1}i$, for all $i = 1, \dots, r$. Hence, condition (A1) holds and $d(\rho_i) \neq 0$ implies that Γ has at least one arrow starting in i . This is one implication in (A2). In order to show the other implication, we assume that Γ has an arrow starting in i . Then the mesh relation m_i starting in i is non-zero. Since m_i is a relation for $\Lambda(\mathcal{T})$, it is generated by $\{d(\rho_1), \dots, d(\rho_r)\}$. In other words, there exists an index set P and elements $c_{pj}, c^{pj} \in \widehat{k\Gamma}$ ($(p, j) \in P \times \{1, \dots, r\}$) such that

$$m_i = \sum_{j=1}^r \sum_{p \in P} c_{pj} d(\rho_j) c^{pj}. \quad (4.10)$$

Let J be the ideal of $\widehat{k\Gamma}$ generated by all arrows. Since B^* is a minimal A_∞ -algebra, it follows that $d(\rho_j) \in J^2$ holds for any $j = 1, \dots, r$, see Section 2.13. If $j \neq i$ and $c_{pj} d(\rho_j) c^{pj} \neq 0$, then $c_{pj} d(\rho_j) c^{pj}$ is a combination of paths of length at least 4, because m_i is a combination of paths from i to $\tau^{-1}i$, while $d(\rho_j)$ is a combination of path from j to $\tau^{-1}j$. Since m_i is a combination of paths of length 2, (4.10) implies that $\sum_{p \in P} c_{pi} d(\rho_i) c^{pi}$ is non-zero and its length 2 component equals m_i . In particular, $d(\rho_i)$ is non-zero and cannot be generated by $\{d(\rho_j)\}_{j \neq i}$. To summarise, $d(\rho_i) \neq 0$ if and only if Γ has arrows starting in i (A2), and the non-zero $d(\rho_i)$'s form a set of minimal relations for $\Lambda(\mathcal{T})$ (A3). The proof is complete. \square

Remark 4.9. Let \mathcal{T} be an idempotent complete Hom-finite algebraic triangulated category with only finitely many isomorphism classes of indecomposable objects. We say that \mathcal{T} is *standard* if the Auslander algebra $\Lambda(\mathcal{T})$ is given by the Auslander–Reiten quiver with mesh relations, see [1, Section 5]. Examples of non-standard categories can be found in [81, 8].

Assume that \mathcal{T} is standard and $\mathcal{T} \cong \underline{\mathcal{E}}$ for some Frobenius category \mathcal{E} satisfying (FM1)–(FM4). Theorem 4.7 shows that up to direct summands $\Delta_{\mathcal{E}}(A)$ is determined by the Auslander–Reiten quiver of \mathcal{T} .

5. MAXIMAL COHEN–MACAULAY MODULES OVER GORENSTEIN RINGS

Let k be an algebraically closed field. Throughout this subsection (R, \mathfrak{m}) and (R', \mathfrak{m}') denote commutative local complete Gorenstein k -algebras, such that their respective residue fields are isomorphic to k .

The results in this section actually hold in greater generality. Namely, we may (at least) replace R and R' respectively by Gorenstein S -orders in the sense of [10, Section III.1] or finite dimensional selfinjective k -algebras. Here, $S = (S, \mathfrak{n})$ denotes a complete regular Noetherian k -algebra, with $k \cong S/\mathfrak{n}$. We decided to stay in the more restricted setup above to keep the exposition clear and concise. It is mostly a matter of heavier notation and not hard to work out the more general results.

5.1. Classical singularity categories. Let $\mathbf{MCM}(R)$ be the category of maximal Cohen–Macaulay R -modules. Note, that $\mathbf{MCM}(R)$ is a Frobenius category with $\mathbf{proj} \mathbf{MCM}(R) = \mathbf{proj} -R$ (see e.g. [22]). Hence, $\underline{\mathbf{MCM}}(R) = \mathbf{MCM}(R)/\mathbf{proj} -R$ is a triangulated category [41].

The following concrete examples of hypersurface rings are of particular interest: Let $R = \mathbb{C}[[z_0, \dots, z_d]]/(f)$, where $d \geq 1$ and f is one of the following polynomials

$$\begin{aligned} (A_n) \quad & z_0^2 + z_1^{n+1} + z_2^2 + \dots + z_d^2 \quad (n \geq 1), \\ (D_n) \quad & z_0^2 z_1 + z_1^{n-1} + z_2^2 + \dots + z_d^2 \quad (n \geq 4), \\ (E_6) \quad & z_0^3 + z_1^4 + z_2^2 + \dots + z_d^2, \\ (E_7) \quad & z_0^3 + z_0 z_1^3 + z_2^2 + \dots + z_d^2, \\ (E_8) \quad & z_0^3 + z_1^5 + z_2^2 + \dots + z_d^2. \end{aligned}$$

Such a \mathbb{C} -algebra R is called *ADE-singularity* of dimension d . As hypersurface singularities they are known to be Gorenstein (see e.g. [21]).

Theorem 5.2 ([58, 85]). *Let $d \geq 1$. Let $S = k[[z_0, \dots, z_d]]$ and $f \in (z_0, \dots, z_d)$. Set $R = S/(f)$ and $R' = S[[x, y]]/(f + xy)$. Then there is a triangle equivalence*

$$\underline{\mathbf{MCM}}(R') \rightarrow \underline{\mathbf{MCM}}(R). \quad (5.1)$$

Definition 5.3. We say that R is *MCM-finite* if there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay R -modules.

In particular, Knörrer’s Periodicity Theorem 5.2 shows that R is *MCM-finite* if and only if $R' = S[[x, y]]/(f + xy)$ is *MCM-finite*. Since the ADE-curve and surface singularities are known to be *MCM-finite* by work of Kiyek & Steinke [57] respectively Artin & Verdier [7], one obtains the following:

Corollary 5.4. *Let R be an ADE-singularity as above. Then R is MCM-finite.*

Remark 5.5. If k is an arbitrary algebraically closed field, then the ADE-polynomials listed above still describe *MCM-finite* singularities. Yet there exist further *MCM-finite* rings if k has characteristic 2, 3 or 5 (complete lists are contained in [38]).

5.6. Relative singularity categories. Henceforth, let F' be a finitely generated R -module and $F = R \oplus F'$. We call $A = \mathbf{End}_R(F)$ a *partial resolution* of R . If A has finite global dimension we say that A is a *resolution*. Denote by $e \in A$ the idempotent endomorphism corresponding to the identity morphism $\mathbf{1}_R$ of R .

The situation is particularly nice if R is *MCM-finite*. Let $M_0 = R, M_1, \dots, M_t$ be representatives of the indecomposable objects of $\mathbf{MCM}(R)$. Their endomorphism

algebra $\text{Aus}(\text{MCM}(R)) = \text{End}_R(\bigoplus_{i=0}^t M_i)$ is called the *Auslander algebra*. Iyama [44] has shown that its global dimension is bounded above by the Krull dimension of R (respectively by 2 in Krull dimensions 0 and 1; for this case see also Auslander's treatment in [9, Sections III.2 and III.3]). Hence, $\text{Aus}(\text{MCM}(R))$ is a resolution of R .

The next lemma motivates the definition of the *relative singularity categories*.

Lemma 5.7. *There is a fully faithful triangle functor $K^b(\text{proj } -R) \rightarrow \mathcal{D}^b(\text{mod } -A)$.*

Proof. The definition of F yields an additive embedding $\text{proj } -R \subseteq \text{add}_R F$. Moreover, there is an additive equivalence $\text{Hom}_R(F, ?): \text{add}_R(F) \rightarrow \text{proj } -A$. Composing these functors and passing to the homotopy categories yields an embedding of triangulated categories $K^b(\text{proj } -R) \rightarrow K^b(\text{proj } -A) \subseteq \mathcal{D}^b(\text{mod } -A)$. \square

Definition 5.8. In the notations above and using Lemma 5.7 we can define the *relative singularity category* of the pair (R, A) as the triangulated quotient category

$$\Delta_R(A) = \frac{\mathcal{D}^b(\text{mod } -A)}{K^b(\text{proj } -R)}. \quad (5.2)$$

Remark 5.9. Using the A -isomorphism $\text{Hom}_R(F, R) \cong eA$, we may rewrite (5.2) as

$$\Delta_R(A) \cong \mathcal{D}^b(\text{mod } -A) / \text{thick}(eA) \quad (5.3)$$

We will use both presentations of $\Delta_R(A)$ in the sequel. Since eA is a projective A -module, $\Delta_R(A)$ is a relative singularity category in the sense of Chen [26]. Different notions of relative singularity categories were introduced and studied by Positselski [76] and also by Burke & Walker [24]. We thank Greg Stevenson for bringing this unfortunate coincidence to our attention.

Let G' be another finitely generated R -module, which contains F' as a direct summand. As above, we define $G = R \oplus G'$, $A' = \text{End}_R(G)$ and $e' = \mathbf{1}_R \in A'$.

We compare the relative singularity categories of A and A' respectively.

Proposition 5.10. *If A is a resolution then there is a fully faithful triangle functor*

$$\Delta_R(A) \longrightarrow \Delta_R(A'). \quad (5.4)$$

Proof. By definition of G' there is an inclusion $\text{add } F \subseteq \text{add } G$. Hence, applying the additive equivalences $\text{Hom}_R(F, ?)$ and $\text{Hom}_R(G, ?)$ respectively we obtain an inclusion $\text{proj } -A \subseteq \text{proj } -A'$. This yields a triangle embedding $K^b(\text{proj } -A) \subseteq K^b(\text{proj } -A')$. Since A has finite global dimension $K^b(\text{proj } -A) \cong \mathcal{D}^b(\text{mod } -A)$ holds. We obtain a triangle embedding $\iota: \mathcal{D}^b(\text{mod } -A) \rightarrow \mathcal{D}^b(\text{mod } -A')$. One checks that $\iota(\text{thick}(eA)) \cong \text{thick}(e'A')$. Now, taking quotients completes the proof. \square

Remark 5.11. The assumption on the global dimension of A is necessary. As an example consider the nodal curve singularity $A = R = k[[x, y]]/xy$ and its Auslander algebra $A' = \text{End}_R(R \oplus k[[x]] \oplus k[[y]])$. In this situation Proposition 5.10 would yield an embedding $\underline{\text{MCM}}(R) = \Delta_R(R) \rightarrow \Delta_R(A')$. But, $\underline{\text{MCM}}(R)$ contains an indecomposable object X with $X \cong X[2s]$ for all $s \in \mathbb{Z}$. Whereas, $\Delta_R(A')$ does not contain such objects by the explicit description obtained in [23, Section 4]. Contradiction.

Without restriction we may assume that F' has no projective direct summands.

Proposition 5.12. *There exists an equivalence of triangulated categories*

$$\frac{\Delta_R(A)}{\mathcal{D}_{A/AeA}^b(\text{mod} - A)} \longrightarrow \underline{\text{MCM}}(R). \quad (5.5)$$

Proof. Buchweitz has shown that there exists an equivalence of triangulated categories $\underline{\text{MCM}}(R) \cong \mathcal{D}_{sg}(R)$ ([22]). We have an isomorphism of rings $R \cong eAe$. Hence, the special case $f = e$ of Proposition 3.3 yields a triangle equivalence

$$\frac{\mathcal{D}^b(\text{mod} - A)/\text{thick}(eA)}{\text{thick}(q(\text{mod} - A/AeA))} \longrightarrow \mathcal{D}_{sg}(R). \quad (5.6)$$

It remains to note that $\text{thick}_{\Delta_R(A)}(q(\text{mod} - A/AeA)) \cong \text{thick}_{\mathcal{D}^b(\text{mod} - A)}(\text{mod} - A/AeA)$, since there are no non-trivial morphisms from $\text{thick}(eA)$ to $\text{thick}(\text{mod} - A/AeA)$. \square

We want to give an intrinsic description of the full subcategory $\mathcal{D}_{A/AeA}^b(\text{mod} - A)$ inside the relative singularity category $\Delta_R(A)$. We need some preparation.

Proposition 5.13. *In the notations of Propositions 2.16 and 5.12 assume additionally that A has finite global dimension and A/AeA is finite dimensional.*

Then there exists a non-positive dg algebra B and a commutative diagram

$$\begin{array}{ccccc} \text{thick}_{\mathcal{D}^b(\text{mod} - A)}(\text{mod} - A/AeA) & \hookrightarrow & \Delta_R(A) & \twoheadrightarrow & \underline{\text{MCM}}(R) \\ \cong \uparrow i_* & & \cong \downarrow i^* & & \cong \downarrow \mathbb{I} \\ \mathcal{D}_{fd}(B) & \hookrightarrow & \text{per}(B) & \twoheadrightarrow & \text{per}(B)/\mathcal{D}_{fd}(B) \end{array} \quad (5.7)$$

where the horizontal arrows denote (functors induced by) the canonical inclusions and projections respectively. Finally, the triangle functor \mathbb{I} is induced by i^* .

Proof. Firstly, $\Delta_R(A)$ is idempotent complete: using Schlichting's negative K-Theory for triangulated categories [82] this may be deduced from the idempotent completeness of $\underline{\text{MCM}}(R)$ (see [23, Theorem 3.2.]). Since A has finite global dimension Corollary 2.18 implies the existence of a dg k -algebra B with $i^*: \mathcal{D}^b(\text{mod} - A)/\text{thick}(eA) \cong \text{per}(B)$. Moreover, since $\dim_k(A/AeA)$ is finite $i_*: \mathcal{D}_{fd}(B) \cong \text{thick}(\text{mod} - A/AeA)$ by the same corollary. The inclusion $\text{thick}_{\mathcal{D}^b(\text{mod} - A)}(\text{mod} - A/AeA) \hookrightarrow \Delta_R(A)$ is induced by the inclusion $\text{mod} - A/AeA \hookrightarrow \text{mod} - A$ (see the proof of Proposition 5.12). Since i_* and i^* are part of a recollement (Proposition 2.16) we obtain $i^* \circ i_* = \mathbf{1}_{\mathcal{D}(B)}$. Hence, the first square commutes. The second square commutes by definition of \mathbb{I} . \square

Note that under the assumptions of Proposition 5.13, we have equalities

$$\mathcal{D}_{fd, A/AeA}(A) = \text{thick}_{\mathcal{D}^b(\text{mod} - A)}(\text{mod} - A/AeA) = \mathcal{D}_{A/AeA}^b(\text{mod} - A). \quad (5.8)$$

Moreover, combining this Proposition with Proposition 2.10 yields the following.

Proposition 5.14. *In the setup of Prop. 5.13, the category $\Delta_R(A)$ is Hom-finite.*

Definition 5.15. For a triangulated k -category \mathcal{T} the full triangulated subcategory

$$\mathcal{T}_{hf} = \left\{ X \in \mathcal{T} \mid \dim_k \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(Y, X[i]) < \infty \text{ for all } Y \in \mathcal{T} \right\}$$

is called *subcategory of right homologically finite objects*.

Example 5.16. If B is a dg k -algebra satisfying $\mathcal{D}_{fd}(B) \subseteq \mathrm{per}(B)$, then $\mathrm{per}(B)_{hf} = \mathcal{D}_{fd}(B)$. Indeed, this follows from $\mathrm{Hom}(B, X[i]) \cong H^i(X)$ for any dg B -module X .

Corollary 5.17. *In the notations of Proposition 5.13 there is an equality*

$$\mathcal{D}_{A/AeA}^b(\mathrm{mod} - A) = \Delta_R(A)_{hf}. \quad (5.9)$$

Proof. This follows from Proposition 5.13 in conjunction with Example 5.16. \square

5.18. Main result. Now, we are able to state and prove the main result of this article. In particular, it applies to the ADE-singularities, which are listed above.

Theorem 5.19. *If R and R' are MCM-finite and $A = \mathrm{Aus}(\mathrm{MCM}(R))$ respectively $A' = \mathrm{Aus}(\mathrm{MCM}(R'))$ denote the Auslander algebras, then the following are equivalent.*

- (a) *There exists an additive equivalence $\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MCM}}(R')$, which respects the action of the respective Auslander–Reiten translations on objects.*
- (b) *There is an equivalence $\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MCM}}(R')$ of triangulated categories.*
- (c) *There exists a triangle equivalence $\Delta_R(A) \cong \Delta_{R'}(A')$.*

Moreover, the implication $[(c) \Rightarrow (b)]$ (and hence also $[(c) \Rightarrow (a)]$) holds under much weaker assumptions. Namely, if A and A' are non-commutative resolutions of isolated Gorenstein singularities R and R' respectively.

Proof. $[(b) \Rightarrow (a)]$ Clear.

$[(a) \Rightarrow (c)]$ Let R be MCM-finite. It is sufficient to show that the Frobenius category $\mathrm{MCM}(R)$ satisfies the assumptions of Theorem 4.7. Indeed, this implies

$$\Delta_R(A) \cong \mathrm{per}(\Lambda_{dg}(\underline{\mathrm{MCM}}(R))), \quad (5.10)$$

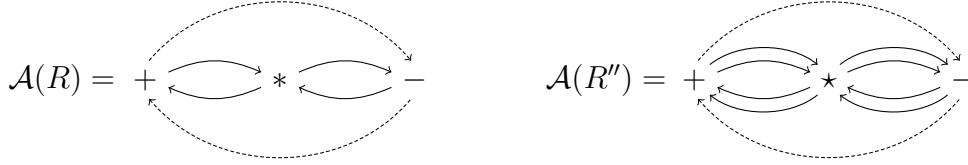
since $\Delta_R(A)$ has split idempotents ([23, Theorem 3.2]). But by construction the dg Auslander algebra $\Lambda_{dg}(\underline{\mathrm{MCM}}(R))$ only depends on the additive structure of $\underline{\mathrm{MCM}}(R)$ and the action of its Auslander–Reiten translation on objects. The claim follows.

The assumptions, which we have to verify are: existence of almost split sequences in $\mathrm{MCM}(R)$; Hom-finiteness and idempotent completeness of the stable category $\underline{\mathrm{MCM}}(R)$. The last property follows from idempotent completeness of $\mathrm{MCM}(R)$ and the existence of lifts of idempotents from $\underline{\mathrm{MCM}}(R)$ to $\mathrm{MCM}(R)$, which holds since R is complete. The first two assertions were shown by M. Auslander. Precisely, in our situation R -lattices (cf. [11, Appendix]) are Cohen–Macaulay, hence MCM-finiteness and [11, Corollary A.2] imply that R is an isolated singularity. Then the notions of Cohen–Macaulay R -modules and R -lattices coincide. Now, the main theorem in *op. cit.* completes the proof.

[(c) \Rightarrow (b)] We claim that this is a consequence of Proposition 5.12 and Corollary 5.17. Indeed, by Proposition 5.12 the stable category $\underline{\mathbf{MCM}}(R)$ is a quotient of $\Delta_R(A)$ and by Corollary 5.17 the kernel of the quotient functor $\mathcal{D}_{A/AeA}^b(\mathbf{mod} - A) \subseteq \Delta_R(A)$ has an intrinsic characterization. Hence, the triangle equivalence in (c) induces an equivalence between the respective quotient categories as in (b).

We verify the (stronger) assumptions of Corollary 5.17. **Hom-finiteness** of $\underline{\mathbf{MCM}}(R)$ follows as in the proof of [(a) \Rightarrow (c)] and holds more generally for any (complete) isolated singularity R . In particular, the algebra A/AeA is finite dimensional. The Auslander algebra of $\mathbf{MCM}(R)$ has finite global dimension by [44, Section 4.2.3]. \square

Example 5.20. Let $R = \mathbb{C}[[u, v]]/(uv)$ and $R'' = \mathbb{C}[[u, v, w, x]]/(uv + wx)$ be the one and three dimensional A_1 -singularities, respectively. The latter is also known as the “conifold”. The Auslander–Reiten quivers $\mathcal{A}(R)$ and $\mathcal{A}(R'')$ of $\mathbf{MCM}(R)$ respectively $\mathbf{MCM}(R'')$, are known, cf. [83] (in particular, [83, Remark 6.3] in dimensions ≥ 3):



Let A and A'' be the respective Auslander algebras of $\mathbf{MCM}(R)$ and $\mathbf{MCM}(R'')$. They are given as quivers as quivers with relations, where the quivers are just the “solid” subquivers of $\mathcal{A}(R)$ and $\mathcal{A}(R'')$, respectively. Now, Knörrer’s Periodicity Theorem 5.2 and Theorem 5.19 above show that there is an equivalence of triangulated categories

$$\frac{\mathcal{D}^b(\mathbf{mod} - A)}{K^b(\mathbf{add} P_*)} \longrightarrow \frac{\mathcal{D}^b(\mathbf{mod} - A'')}{K^b(\mathbf{add} P_\star)}, \quad (5.11)$$

where P_* is the indecomposable projective A -module corresponding to the vertex $*$ and similarly $P_\star \in \mathbf{proj} - A''$ corresponds to \star .

Note, that the relative singularity category $\Delta_R(A) = \mathcal{D}^b(\mathbf{mod} - A)/K^b(\mathbf{add} P_*)$ from above has an explicit description, see [23, Section 4].

Remark 5.21. For finite dimensional selfinjective k -algebras of finite representation type one can prove (the analogue of) implication [(b) \Rightarrow (c)] in Theorem 5.19 above without relying on dg-techniques. Indeed, Asashiba [8, Corollary 2.2.] has shown that in this context stable equivalence implies derived equivalence. Now, Rickard’s [79, Corollary 5.5.] implies that the respective Auslander algebras are derived equivalent (a result, which was recently obtained by W. Hu and C.C. Xi in a much more general framework [43, Corollary 3.13]¹). One checks that this equivalence induces a triangle equivalence between the respective relative singularity categories. This result is stronger than the analogue of Theorem 5.19 (c).

¹The first author would like to thank Sefi Ladkani for pointing out this reference.

5.22. Global relative singularity categories. Let X be a quasi-projective scheme and \mathcal{F} a coherent sheaf, which is locally free on $X \setminus \text{Sing}(X)$. We assume that $\mathcal{A} = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{F})$ has finite global dimension. Hence, the ringed space $\mathbb{X} = (X, \mathcal{A})$ is a non-commutative resolution of X and $\mathcal{D}^b(\text{Coh}(\mathbb{X}))$ is a categorical resolution in the spirit of works of Van den Bergh [87], Kuznetsov [60] and Lunts [65]. There is a triangle embedding $\text{Perf}(X) \rightarrow \mathcal{D}^b(\text{Coh}(\mathbb{X}))$. Thus, we can define the *relative singularity category* as the idempotent completion [14] of the corresponding triangulated quotient category: $\Delta_X(\mathbb{X}) = (\mathcal{D}^b(\text{Coh}(\mathbb{X}))/\text{Perf}(X))^\omega$. If X has *isolated* singularities, then the study of $\Delta_X(\mathbb{X})$ reduces to the “local” relative singularity categories defined above. Precisely, there exists an triangle equivalence [23, Cor. 2.11.]

$$\Delta_X(\mathbb{X}) \cong \bigoplus_{x \in \text{Sing}(X)} \Delta_{\hat{\mathcal{O}}_x}(\hat{\mathcal{A}}_x). \quad (5.12)$$

If X is a curve with nodal singularities, then this yields a complete and explicit description of the category $\Delta_X(\mathbb{X})$, where \mathcal{A} is the *Auslander sheaf* of X [23].

6. REMARKS ON RELATED WORK

6.1. Relationship to Bridgeland’s moduli space of stability conditions. Let $X = \text{Spec}(R_Q)$ be a Kleinian singularity with *minimal* resolution $f: Y \rightarrow X$ and exceptional divisor $E = f^{-1}(0)$. Then E is a tree of rational (-2) -curves, whose dual graph Q is of ADE-type. Let us consider the following triangulated category

$$\mathcal{D} = \ker(\mathbb{R}f_*: \mathcal{D}^b(\text{Coh}(Y)) \rightarrow \mathcal{D}^b(\text{Coh}(X))). \quad (6.1)$$

Bridgeland determined a connected component $\text{Stab}^\dagger(\mathcal{D})$ of the stability manifold of \mathcal{D} [19]. More precisely, he proves that $\text{Stab}^\dagger(\mathcal{D})$ is a covering space of $\mathfrak{h}^{\text{reg}}/W$, where $\mathfrak{h}^{\text{reg}} \subseteq \mathfrak{h}$ is the complement of the root hyperplanes in a fixed Cartan subalgebra \mathfrak{h} of the complex semi-simple Lie algebra \mathfrak{g} of type Q and W is the associated Weyl group. It turns out, that $\text{Stab}^\dagger(\mathcal{D})$ is even a *universal* covering of $\mathfrak{h}^{\text{reg}}/W$. This follows [19] from a faithfulness result for the braid group actions generated by spherical twists (see [84] for type A and [18] for general Dynkin types).

The category \mathcal{D} admits a different description. Namely, as category of dg modules with finite dimensional total cohomology $\mathcal{D}_{fd}(B)$, where $B = B_Q$ is the dg-Auslander algebra $\Lambda_{dg}(\underline{\text{MCM}}(R))$ of $R = \hat{R}_Q$. Let $A = \text{Aus}(\text{MCM}(R))$ be the Auslander algebra of $\text{MCM}(R)$ and denote by e the identity endomorphism of R considered as an idempotent in A . Then the derived McKay–Correspondence [50, 20] induces a commutative diagram of triangulated categories and functors, cf. [19, Section 1.1].

$$\begin{array}{ccccc} \mathcal{D} & \xlongequal{\quad} & \ker(\mathbb{R}f_*: \mathcal{D}^b(\text{Coh}(Y)) \rightarrow \mathcal{D}^b(\text{Coh}(X))) & \hookrightarrow & \mathcal{D}_E^b(\text{Coh}(Y)) \\ \downarrow \cong & & \cong \downarrow & & \downarrow \cong \\ \mathcal{D}_{fd}(B) & \xrightarrow{\quad \cong \quad} & \mathcal{D}_{A/AeA}^b(\text{mod } -A) & \hookrightarrow & \mathcal{D}^b(\text{mod } -A). \end{array} \quad (6.2)$$

For the equivalence $\mathcal{D}_{fd}(B) \cong \mathcal{D}_{A/AeA}^b(\text{mod } -A)$, we refer to Proposition 5.13 and (5.8). Moreover, this category is triangle equivalent to the kernel of the quotient functor $\Delta_R(A) \rightarrow \mathcal{D}_{sg}(R)$, see Proposition 5.12.

Remark 6.2. It would be interesting to study Bridgeland's space of stability conditions for the categories $\mathcal{D}_{fd}(B)$ in the case of odd dimensional ADE-singularities R as well! Note that the canonical t -structure on $\mathcal{D}(B)$ restricts to a t -structure on $\mathcal{D}_{fd}(B)$ by Proposition 2.5. Its heart is the finite length category of finite dimensional modules over the stable Auslander algebra of $\text{MCM}(R)$.

6.3. Links to generalized cluster categories. Let k be an algebraically closed field of characteristic 0. Let Q be a quiver of ADE-type. As above, we consider the dg Auslander algebra $B_Q = \Lambda_{dg}(\underline{\text{MCM}}(\widehat{R}_Q))$ of the corresponding ADE-singularity \widehat{R}_Q of even Krull dimension. There exists an isomorphism of dg algebras

$$B_Q \cong \Pi(Q, 2, 0), \quad (6.3)$$

where $\Pi(Q, d, W)$ denotes the deformed dg preprojective algebra, which was associated to a finite (graded) quiver Q , a positive integer d and a potential W of degree $-d + 3$ by Ginzburg [37] (see also [89]).

B_Q is a bimodule 2-Calabi-Yau algebra in the sense of [37]. Hence, the triangle equivalence (1.6) yields the well-known result that $\underline{\text{MCM}}(\widehat{R}_Q)$ is the 1-cluster category of kQ (see e.g. Reiten [77]). More generally, Van den Bergh's [89, Theorem 10.2.2] shows that $\Pi(Q, d, W)$ is bimodule d -Calabi-Yau².

Now, if $H^0(\Pi(Q, d, W))$ is finite dimensional, then (by definition) the quotient

$$\mathcal{C}_{(Q, d, W)} = \frac{\text{per}(\Pi(Q, d, W))}{\mathcal{D}_{fd}(\Pi(Q, d, W))} \quad (6.4)$$

is a generalized $(d-1)$ -cluster category. In particular, $\mathcal{C}_{(Q, d, W)}$ is $(d-1)$ -Calabi-Yau and the image of $\Pi(Q, d, W)$ defines a $(d-1)$ -cluster tilting object [2][39].

The following Morita-type question attracted a lot of interest recently.

Question 6.4. *Let \mathcal{C} be a k -linear Hom-finite d -Calabi-Yau algebraic triangulated category with d -cluster-tilting object. Is there a triple (Q, d, W) as above such that \mathcal{C} is triangle equivalent to the corresponding cluster category $\mathcal{C}_{(Q, d, W)}$?*

In a recent series of papers Amiot *et. al.* answer this question to the affirmative in some interesting special cases [2, 6, 5, 4]. In [3] Amiot gives a nice overview.

Let us outline another promising approach [49] to tackle Question 6.4: a combination of Keller & Vossieck's [55, Exemple 2.3] with the theory developed in this article shows that for many interesting algebraic triangulated categories \mathcal{T} there exists a

²More precisely, Van den Bergh proves that $\Pi(Q, d, W)$ is *strongly* bimodule Calabi-Yau, which implies the bimodule Calabi-Yau property.

non-positive dg algebra B and a triangle equivalence generalizing (1.6) above

$$\mathcal{T} \cong \frac{\text{per}(B)}{\mathcal{D}_{fd}(B)}. \quad (6.5)$$

In particular, this holds for stable categories of maximal Cohen–Macaulay modules over certain Iwanaga–Gorenstein rings and the Calabi–Yau categories arising from subcategories of nilpotent representations over preprojective algebras (*cf.* [13, 36]). Palu [73] also obtained such an equivalence (in a slightly different form) in his study of Grothendieck groups of Calabi–Yau categories with cluster-tilting objects.

Now, if \mathcal{T} is d –Calabi–Yau category as in Question 6.4, then $\mathcal{D}_{fd}(B)$ is a $(d+1)$ –Calabi–Yau category by Keller & Reiten’s [54, Theorem 5.4]. In conjunction with Van den Bergh’s [89, Theorem 10.2.2], we see that Question 6.4 has an affirmative answer, if the following statement holds (we use the terminology from [89]).

If A is a pseudo-compact dg algebra such that $\mathcal{D}_{fd}(A)$ is a d –Calabi–Yau triangulated category generated by a finite number of simple dg A -modules, then A is a strongly d –Calabi–Yau dg algebra.

This statement has a conjectural status in general. However, for some interesting d –Calabi–Yau categories \mathcal{T} (with d –cluster-tilting object) one can show that B is strongly d –Calabi–Yau without relying on the statement above. For example, this was done by Thanhoffer de Völcsy & Van den Bergh for $\mathcal{T} = \underline{\text{MCM}}(R)$, where R is a complete Gorenstein quotient singularity of Krull dimension three [30]. They also prove (6.5) in a more restricted setup.

7. APPENDIX: DG-AUSLANDER ALGEBRAS FOR ADE–SINGULARITIES

The stable Auslander–Reiten quivers for the curve and surface singularities of Dynkin type ADE are known, see [32] and [12] respectively. Hence, the stable Auslander–Reiten quiver for any ADE–singularity R is known by Knörrer’s periodicity (Theorem 5.2). The equivalence (5.10) in the proof of Theorem 5.19 describes the triangulated category $\Delta_R(\text{Aus}(R))$ as the perfect category for the dg-Auslander algebra associated to $\underline{\text{MCM}}(R)$. We list the graded quivers³ of these dg-algebras for the ADE–singularities in Subsections 7.2 – 7.8. For surfaces, this also follows from [30, 4].

Remark 7.1. For ADE–singularities R , it is well-known that the stable categories $\underline{\text{MCM}}(R)$ are *standard*, i.e. the mesh relations form a set of minimal relations for the Auslander algebra $\text{Aus}(\underline{\text{MCM}}(R))$ of $\underline{\text{MCM}}(R)$ (*cf.* [1, 80], respectively [45]). Hence, the graded quivers completely determine the dg Auslander algebras in this case.

The conventions are as follows. Solid arrows \longrightarrow are in degree 0, whereas broken arrows $- - \rightarrow$ are in degree -1 and correspond to the action of the Auslander–Reiten translation. The differential d is uniquely determined by sending each broken arrow ρ to the mesh relation starting in $s(\rho)$. If there are no irreducible maps

³M.K. thanks Hanno Becker for his help with the TikZ–package.

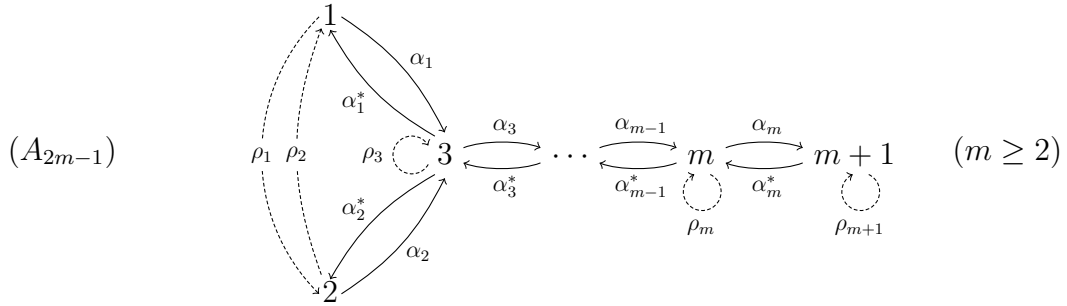
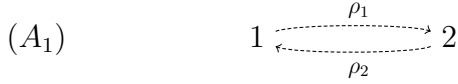
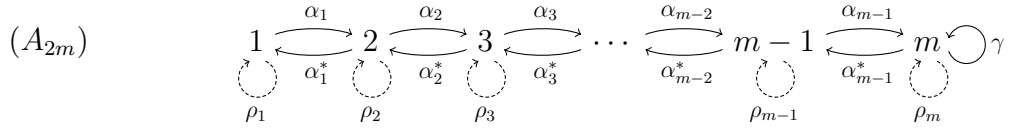
(i.e. solid arrows) starting in the vertex $s(\rho)$, then we set $d(\rho) = 0$ (*cf.* the case of type (A_1) in odd dimension in Subsection 7.2). Let us illustrate this by means of two examples: in type (A_{2m}) in odd Krull dimension (see Subsection 7.2) we have

$$d(\rho_2) = \alpha_1 \alpha_1^* + \alpha_2^* \alpha_2, \quad (7.1)$$

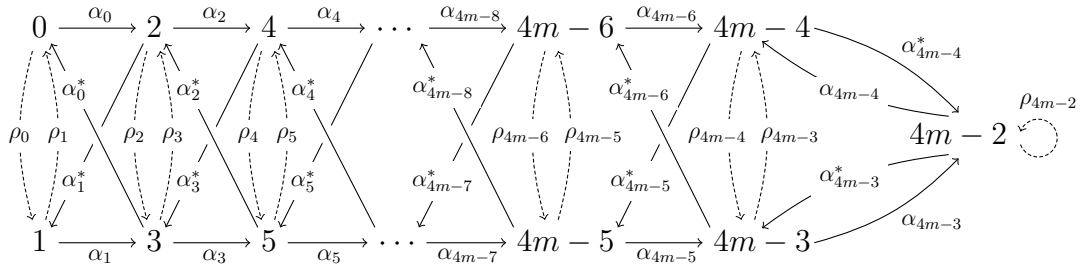
whereas in odd dimensional type (E_8) (see Subsection 7.7)

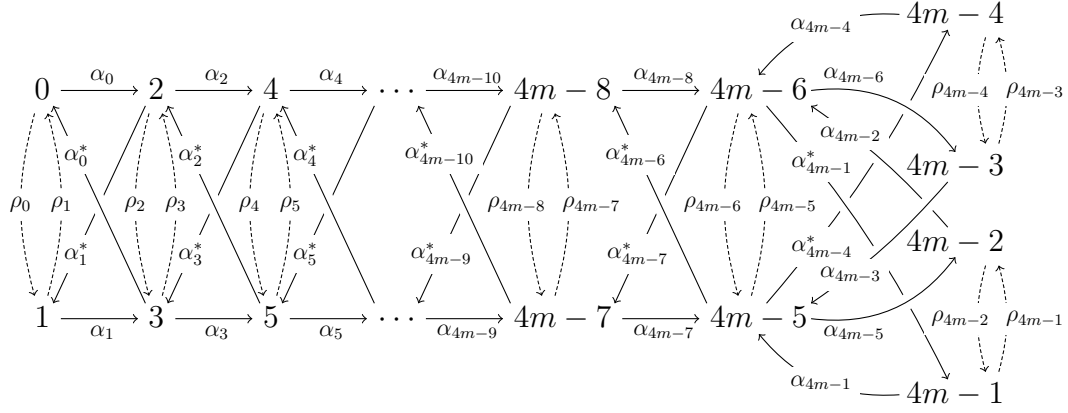
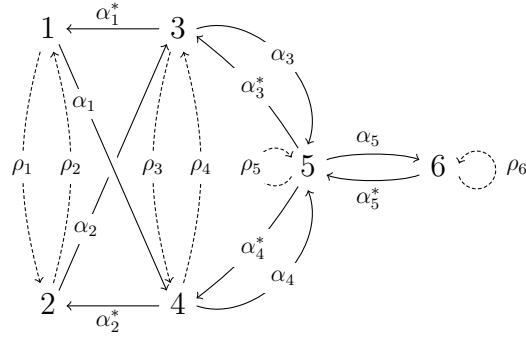
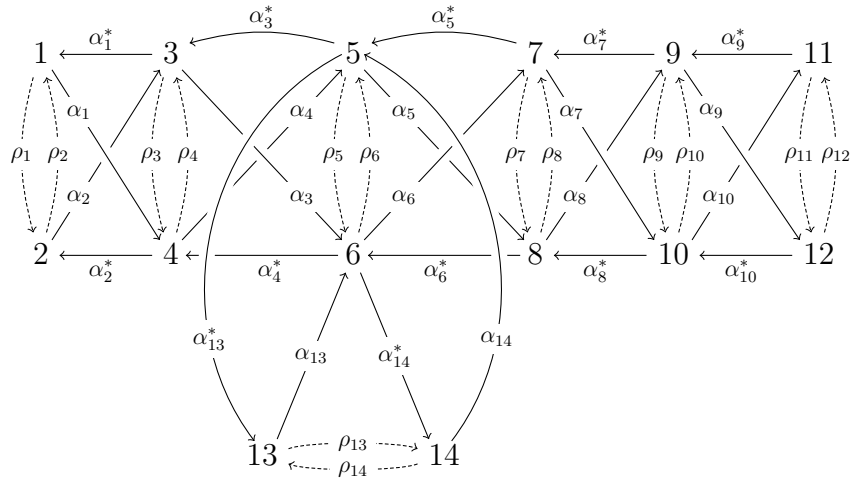
$$d(\rho_{10}) = \alpha_8 \alpha_8^* + \alpha_{16} \alpha_{16}^* + \alpha_9^* \alpha_{10}. \quad (7.2)$$

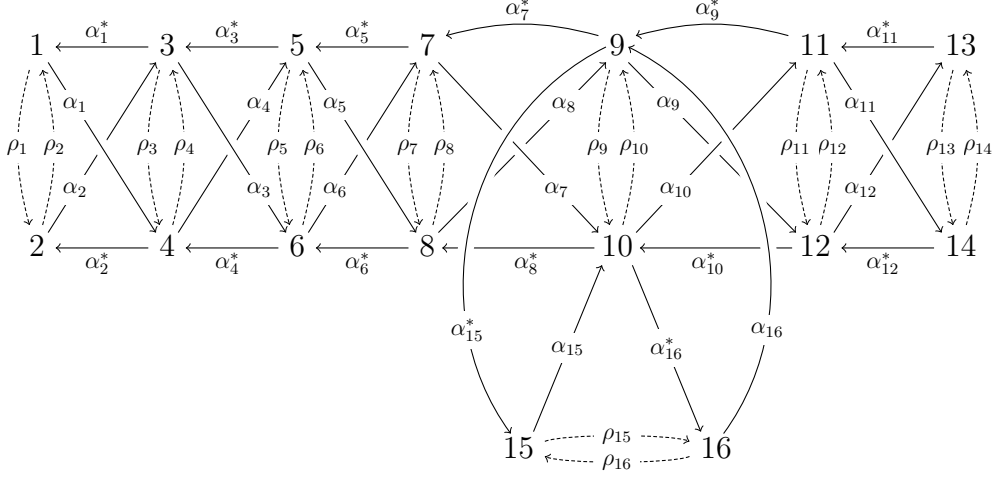
7.2. DG-Auslander algebras for Type A -singularities in odd dimension.



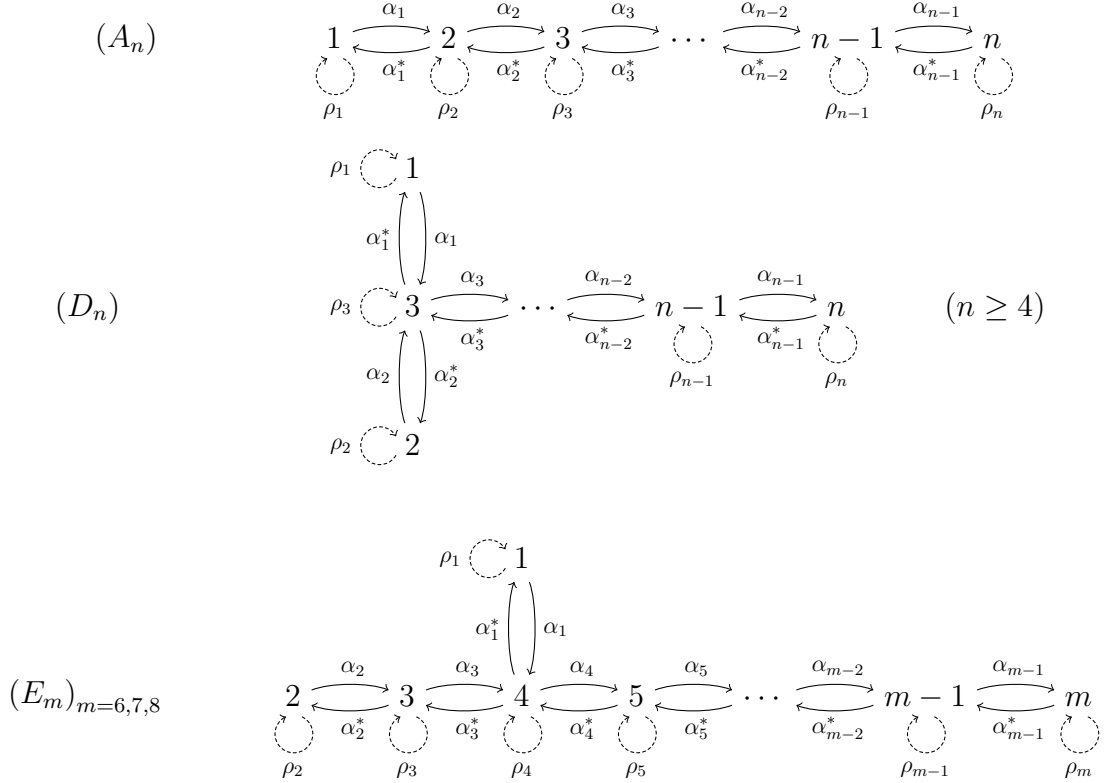
7.3. DG-Auslander algebras of odd dim. (D_{2m+1}) -singularities, $m \geq 2$.



7.4. DG–Auslander algebras of odd dimensional (D_{2m}) -singularities, $m \geq 2$.7.5. The DG–Auslander algebra of odd dimensional (E_6) -singularities.7.6. The DG–Auslander algebra of odd dimensional (E_7) -singularities.

7.7. The DG–Auslander algebra of odd dimensional (E_8) -singularities.

7.8. DG–Auslander algebras of even dimensional ADE–singularities.



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